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To cite this version:

Salvatore Federico, Huyen Pham. Characterization of the optimal boundaries in reversible investment problems. 2013. hal-00676352

HAL Id: hal-00676352

https://hal.science/hal-00676352

Preprint submitted on 4 Jul 2013

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Characterization of the optimal boundaries in reversible investment problems

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This version: July 2013

Abstract

This paper studies a *reversible* investment problem where a social planner aims to control its capacity production in order to fit optimally the random demand of a good. Our model allows for general diffusion dynamics on the demand as well as general cost functional. The resulting optimization problem leads to a degenerate two-dimensional bounded variation singular stochastic control problem, for which explicit solution is not available in general and the standard verification approach can not be applied a priori. We use a direct viscosity solutions approach for deriving some features of the optimal free boundary function, and for displaying the structure of the solution. In the quadratic cost case, we are able to prove a smooth-fit C^2 property, which gives rise to a full characterization of the optimal boundaries and value function.

Keywords: Singular Stochastic Control, Optimal capacity, Reversible Investment, Viscosity solution, Smooth-fit.

AMS Classification: 93E20, 49J40, 49L25.

*Part of this research was done when this author was Post-doc at LPMA, Université Paris Diderot, supported by Alma Research.

1 Introduction

We are concerned with a bounded variation singular control problem motivated by a model of reversible investment. More precisely, we imagine to deal with a social planner whose objective is to optimize some functional depending on the current demand of a good (energy, electricity, oil, corn, etc) and its supply in terms of production capacity that can be increased or decreased at any time and at given proportional costs.

Problems of investment under uncertainty have been introduced in the economic literature by [33] and then developed by several other authors (see [16, Ch. 11] for references on this subject). From a mathematical point of view, such problems have been formulated as optimal stopping problems or, at a second stage of complexity, as singular stochastic optimal control problems, and have given a considerable impulse to the development of the corresponding mathematical theory. As references for the theory of singular stochastic control in context different from investment under uncertainty, we may mention the works [13, 21, 22, 24] and [17, Ch. VIII]. The mathematical literature of singular stochastic control applied to the subject of *irreversible* investment under uncertainty (i.e. when the capacity can be only increased and the control is therefore monotone) includes the works [3, 5, 6, 11, 12, 15, 35, 38, 42]. In particular [6, 38] solve the problem by using a probabilistic representation result stated in [7], which seems very suitable to tackle this kind of problems, while [42] uses a dynamic programming approach. The economic issue of *reversibility* (i.e. when the capacity can be also decreased and the control is a finite variation process) has then been introduced and studied, among others, in [1, 4, 18, 19, 31, 34]. In the papers dealing with reversibility mentioned above, the ones (substantially) considering two state variables (an uncontrolled one containing the noise, and a controlled one, representing the capacity) are [4, 19, 31, 34][1]. [4] derives optimality conditions based on economic considerations, while [19] states and solves the problem with an interesting connection between finite-variation singular control problems and optimal switching problems. The papers dealing with a dynamic programming approach directly on the singular control problem and with the study of the associated Hamilton-Jacobi-Bellman equation (which in this case is a variational inequality) are [31, 34]. In particular, [34] considers an expected performance on infinite horizon with discounting over time, as in our case. However, the approach of [34] is of *verification* type. In a singular stochastic control framework, this means that one has to guess some smooth fit properties of the value function at the optimal free boundary in order to look for a solution of the Hamilton-Jacobi-Bellman equation. Then one needs to prove *a posteriori* that the solution found is indeed the value function and, as a byproduct, one gets also the optimal feedback control. When this approach is applicable, it turns out to be very convenient, as it is theoretically fast (even if it may involve a nontrivial technical complexity) and allows a first understanding of the problem. Moreover, the presence of an explicit solution is an important tool to analyze the qualitative properties of optimal control and trajectory. On the other hand, one has to recognize that it presents two drawbacks. First, it is based on a guess, and so it cannot bring to a deep understanding of the structural issues of the problem. Second, it works only when explicit solutions are available, therefore it leaves the problem completely unsolved most of the

[1] We should mention also [28], which just shows the connection between finite-variation singular control and Dynkin games. We shall indeed use this connection in Subsection 3.2 to prove some results on the value function.

cases.

In the present paper, we perform a direct study of the singular stochastic control problem with bounded variation controls (without passing through verification type arguments) by means of a viscosity approach to the Hamilton-Jacobi-Bellman (HJB) equation. To our knowledge, this is the first time that such an approach is used in the case of two state variables, in particular when the controlled state variable, here the reversible capacity process, has no diffusion term, and so is degenerate[2]. Our approach allows us to keep much more generality with regard to the uncontrolled state variable (which is indeed a very general diffusion in the present paper, as in [4]) and to state the smooth-fit conditions of [34] as *necessary* conditions of optimality, i.e. prove that the value function *must* satisfy these conditions[3]. More precisely, we show that the value function is C^1 along the component of the controlled variable (Proposition 3.1; this easily follows from our assumptions by convexity arguments, just working on the definition of value function). This allows to state the structure of the solution (Theorems 4.1 and 4.2). Then, we prove that it has continuous mixed second derivative along the optimal boundary function (Proposition 5.1; this is a deeper result, which invokes the viscosity property of the value function and requires the additional assumption (5.4) of quadratic cost in the capacity). The set of optimality conditions stated is then rewritten, following the arguments of [4], in a more suitable way, which allows to determine the optimal boundaries, splitting them in three different regions and giving optimality conditions characterizing them in each of these regions (Theorem 5.2). At the end, this machinery allows us to uniquely individuate the value function and solve the problem by Theorem 4.2. We mention that the approach developed in [6] for singular control problem with monotone controls is not valid anymore in the context of reversible investment.

The rest of the paper is organized as follows. In Section 2, we formulate the two-dimensional bounded variation singular stochastic control problem and state the main assumptions. We study in Section 3 some first properties of the value function and of the optimal boundary, which is a function of the demand. In Section 4, by relying on the viscosity property of the value function to its dynamic programming variational inequality, we give a first main result providing the structure of the value function, and state a second main result yielding the optimal control in terms of the optimal boundary. Section 5 focus on the case of quadratic cost function, which allows us to prove a second order smooth fit principle. This leads to the missing information to explicitly individuate the value function and the optimal boundary (the third main result), and

[2] There are of course several papers (among them we may quote [22]), which consider singular stochastic control problems with multidimensional state variables, and characterize the value function in terms of viscosity solutions to the associated HJB equations. However, rather few go beyond the viscosity characterization, and investigate smooth-fit properties in order to derive the structural form of the value function. In this spirit, we may mention the paper [18] in the case of just one dimensional controlled variable. See also [20] for impulse control of multi-dimensional diffusion processes with non degenerate diffusion term. On the other hand, we may quote the paper [40], which studies regularity of a two-dimensional singular control problem with nondegenerate diffusion. Finally, we should mention the paper [41], dealing with a singular control problem with two state variables in a different context (consumption-investment under transaction costs). In this case the problem is approached by dynamic programming and by means of viscosity solutions to the associated Hamilton-Jacobi-Bellman equation. However, the regularity of the value function is proved by reducing the problem to dimension one, which is possible in that case due to the specific structure of the problem.

[3] Another major advantage of such approach is that it allows generalizations. With this regard, we notice that here we minimize a cost functional. However, the arguments used here can be extended to the case of profit/cost functional, as in [34].

makes the results of Section 4 applicable. Finally, we close the paper by explicit illustrations of the theory to the basic example of geometric Brownian motion for the uncontrolled demand diffusion in the case of irreversible investment. More examples and applications are developed, in the case of irreversible investment, in the companion paper [2], where we also take into account delay in the expansion of the capacity production.

2 The singular stochastic control problem

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and supporting a standard one-dimensional Brownian motion $(W_t)_{t\geq 0}$.

On this space, we consider an uncontrolled state process $D = (D_t)_{t\geq 0}$ (representing the demand of a good), governed by a diffusion dynamics:

$$\mathrm{d}D_t \;=\; \mu(D_t)\mathrm{d}t + \sigma(D_t)\mathrm{d}W_t, \quad D_0 = d_0. \tag{2.1}$$

Let

$$\mathcal{O} \;:=\; (d_{\min}, d_{\max}), \quad -\infty \leq d_{\min} < d_{\max} \leq \infty.$$

Throughout the paper we assume the following on the diffusion D.

Assumption 2.1. *(i) The coefficients $\mu, \sigma : \mathcal{O} \to \mathbb{R}$ are continuous and have at most linear growth.*
(ii) For all $d_0 \in \mathcal{O}$, there exists a unique non-exploding solution D^{d_0} admitting a version with continuous path (and we shall always refer to such a version) to the SDE (2.1) *in the space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values into $\mathcal{O}$.*
(iii) The unique solution D continuously depends on the initial datum: if $d_n \overset{n\to\infty}{\longrightarrow} d_0$, then $D^{d_n} \overset{n\to\infty}{\longrightarrow} D^{d_0}$ almost surely.
(iv) The SDE (2.1) *satisfy a comparison criterion: if $d_0 \leq d_0'$, then $D_t^{d_0} \leq D_t^{d_0'}$ $\mathbb{P}$-almost surely for every $t \geq 0$.*
(v) The boundaries $d_{\min}, d_{\max}$ are natural for the diffusion D in the sense of Feller's classification and the diffusion D is regular.

Remark 2.1. Sufficient conditions for the assumptions above can be found in many classical references, such as, e.g., [25, Ch. 5]. We notice that some standard models of diffusion, such as arithmetic or geometric Brownian motion, mean-reverting processes, or the CIR model (for suitable values of the parameters) satisfy Assumption 2.1. □

Next, we denote by $\mathcal{I}$ the class of càdlàg bounded variation $\mathbb{F}$-adapted processes, setting $I_{0^-} = 0$. Given $I \in \mathcal{I}$ we have the minimal decomposition $I = I^+ - I^-$, where I^+, I^- are the positive and the negative variation of I, respectively. It follows that the increments

$$\mathrm{d}I_t^+ \;:=\; I_t^+ - I_{t^-}^+, \qquad \mathrm{d}I^- \;:=\; I_t^- - I_{t^-}$$

are supported on disjoint subsets of $[0, \infty)$. We shall always refer to the latter minimal decomposition and, with a slight abuse of notation, we shall often denote $I = (I^+, I^-)$. The economic meaning of I^+ and I^- is the following:

- I^+_t is the cumulative investment done up to time t to increase the capacity;

- I^-_t is the cumulative disinvestment done up to time t to decrease the capacity.

Hence, the production capacity process $(C_t)_{t\geq 0}$, controlled by $I \in \mathcal{I}$, is given by

$$C_t \quad = \quad c_0 + I^+_t - I^-_t, \qquad c_0 \in \mathbb{R}. \tag{2.2}$$

The objective is to minimize over $\mathcal{I}$

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(g(C_t, D_t)\mathrm{d}t + q_0^+ \mathrm{d}I^+_t + q_0^- \mathrm{d}I^-_t\Big)\Big], \tag{2.3}$$

where $g : \mathbb{R} \times \mathcal{O} \to [0, \infty)$ is a cost function, $q_0^+ > 0$, $q_0^- > 0$ are, respectively, the cost per unit of investment and the cost per unit of disinvestment, and ρ is a positive discount factor.

Remark 2.2. 1. Among all the possible decompositions of a bounded variation process $I \in \mathcal{I}$, the minimal decomposition is the one providing the minimal value for the functional (2.3). Indeed, denoting by $I^{m,+} - I^{m,-}$ the minimal decomposition of I, for all the other decompositions $I = I^+ - I^-$ the dynamics of the capacity C is the same, while $I^+ \geq I^{m,+}$, $I^- \geq I^{m,-}$. So

$$\begin{aligned} & \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(g(C_t, D_t)\mathrm{d}t + q_0^+ \mathrm{d}I^{m,+}_t + q_0^- \mathrm{d}I^{m,-}_t\Big)\Big] \\ \leq \; & \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(g(C_t, D_t)\mathrm{d}t + q_0^+ \mathrm{d}I^+_t + q_0^- \mathrm{d}I^-_t\Big)\Big], \end{aligned}$$

2. Even if we shall consider q_0^- as a finite number, everything can be extended, giving a suitable sense, to the case $q_0^- = \infty$. In this case the problem is equivalent to require irreversibility for the investment (i.e. the case when I^- is constrained to be 0, as there is no convenience to disinvest, the cost being infinite). This case is treated in Subsection 5.3.
3. For sake of simplicity, we do not impose the (economically meaningful: recall that C should represent the capacity production) state constraint $C_t \geq 0$. We will comment in Remark 4.2 about the case that it may be verified a posteriori.
4. Note that, with respect to the usual investment under uncertainty literature, which is mainly based on profit/cost performance criterions, we focus here on the minimization of a cost criterion in the spirit of a social planning problem, whose objective is to fit the capacity production to the demand at cheapest cost. In particular the most significant case from the economic point of view is when $g(c, d) = |c - d|^2$ (see also Remark 2.3 (2) below), as it represents a maximization of social surplus in the context of a linear inverse demand function (see [2] for a detailed description and explanation). We will give an explicit solution to the problem exactly in that case. $\square$

We shall make the following assumptions on the cost function g.

Assumption 2.2. *(i) $g \in C^0(\mathbb{R} \times \mathcal{O}; \mathbb{R}_+)$, $g(\cdot, d) \in C^1(\mathbb{R}; \mathbb{R})$ for every $d \in \mathcal{O}$, and $g_c \in C^0(\mathbb{R} \times \mathcal{O}; \mathbb{R})$.*
(ii) $g(\cdot, d)$ is convex for all $d \in \mathcal{O}$ and $g_c(c, \cdot)$ is nonincreasing in $\mathcal{O}$ for every $c \in \mathbb{R}$.
(iii) g and g_c satisfy a polynomial growth condition w.r.t. d: there exist positive locally bounded functions $\gamma_0, \eta_0 : \mathbb{R} \to \mathbb{R}$, and a constant $\nu \geq 0$ such that

$$|g(c, d)| + |g_c(c, d)| \quad \leq \quad \gamma_0(c) + \eta_0(c)|d|^\nu, \qquad \forall\, c \in \mathbb{R},\ \forall\, d \in \mathcal{O}. \tag{2.4}$$

Remark 2.3. **1.** We observe that the monotonicity assumption in Assumption 2.2-(ii) reflects an economic intuition. It means that the marginal cost with respect to capacity for a fixed level of capacity is nonincreasing in the demand: for a given level of capacity, the more is the demand, the more is convenient to invest; the less is the demand, the more is convenient to disinvest.
2. Any function g of the spread $|c-d|$ between capacity and demand, in the form

$$g(c,d) \;=\; K_0|c-d|^{\alpha}, \quad K_0 \geq 0,\ \alpha > 1, \tag{2.5}$$

satisfies Assumption 2.2. □

Remark 2.4. Following the idea of [5, Sec. 6], our model admits a suitable generalization to the case of capacity dynamics in the form:

$$\mathrm{d}C_t \;=\; C_t(b\,\mathrm{d}t + \gamma\,\mathrm{d}W_t^0) + \mathrm{d}I_t, \;\; C_{0^-} \;=\; c,$$

where W^0 is another Browinan motion independent of W. Indeed letting C^0 be the solution to

$$\mathrm{d}C_t^0 \;=\; C_t^0(b\,\mathrm{d}t + \gamma\,\mathrm{d}W_t^0), \;\; C_0^0 \;=\; 1,$$

the process C can be rewritten as

$$C_t \;=\; C_t^0\bar{C}_t, \qquad t \geq 0,$$

where

$$\bar{C}_t \;=\; c + \bar{I}_t^+ - \bar{I}_t^-, \quad \text{with} \quad \bar{I}_t^+ \;=\; \int_0^t \frac{1}{C_s^0}\,\mathrm{d}I_s^+, \quad \bar{I}_t^- \;=\; \int_0^t \frac{1}{C_s^0}\,\mathrm{d}I_s^-.$$

So, letting $\tilde{g}(\bar{c}, c^0, d) = g(c^0\bar{c}, d)$, the problem becomes

$$\inf_{\bar{I}\in\mathcal{I}} \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(\tilde{g}(\bar{C}_t, C_t^0, D_t)\mathrm{d}t + C_t^0(q_0^+\mathrm{d}\bar{I}_t^+ + q_0^-\mathrm{d}\bar{I}_t^-)\Big)\Big].$$

This problem involves an additional uncontrolled state variable (the variable C^0), but keeps the basic structures, so it seems approachable by the same techniques developed in the next sections. □

3 Dynamic programming: preliminary results

We shall study the optimization problem by dynamic programming methods, and so we consider this singular stochastic control problem when varying initial data $(c_0, d_0) = (c, d) \in \mathbb{R}\times\mathcal{O}$. Therefore, from now on, we stress the dependence of C on c, I and the dependence of D on d by denoting them respectively as $C^{c,I}$, D^d. The state space is then equal to

$$\mathcal{S} \;=\; \mathbb{R}\times\mathcal{O}.$$

Throughout the paper we indicate by $C^{h,k}(\mathcal{S};\mathbb{R})$, $h, k \in \mathbb{N}$, the class of functions which are continuous, h-times differentiable with respect to the first variable, k-times differentiable with respect to the second variable, and having these derivatives continuous in $\mathcal{S}$.

Given $(c,d) \in \mathcal{S}$, the functional to be minimized over $I \in \mathcal{I}$ is

$$G(c,d;I) \;:=\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(g(C_t^{c,I}, D_t^d)\mathrm{d}t + q_0^+ \mathrm{d}I_t^+ + q_0^- \mathrm{d}I_t^-\Big)\Big],$$

and the associated value function is

$$v(c,d) \;:=\; \inf_{I\in\mathcal{I}} G(c,d;I), \quad (c,d)\in\mathcal{S}. \tag{3.1}$$

3.1 First properties of the value function: finiteness and convexity

Notice that $v \geq 0$ as $g \geq 0$. We want to ensure also an upper bound for v. Since μ, σ have at most linear growth, by standard estimates we know (see, e.g., [30, Ch. 2.5, Cor. 12]) that there exist constants $K_0 = K_{0,\mu,\sigma,\nu} \geq 0$ and $K_1 = K_{1,\mu,\sigma,\nu} \in \mathbb{R}$ such that

$$\mathbb{E}\Big[\big|D_t^d\big|^\nu\Big] \;\leq\; K_0(1+|d|^\nu)e^{K_1 t}, \quad \forall t \geq 0. \tag{3.2}$$

In the sequel, we make the standing assumption that the discount factor ρ satisfies

$$\rho \;>\; K_1^+, \tag{3.3}$$

where K_1 is the constant appearing in (3.2). Using Assumption 2.2 (iii) and (3.2)-(3.3), we get

$$\hat{V}(c,d) \;:=\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t} g(c, D_t^d)\mathrm{d}t\Big] \;\leq\; \gamma_1(c) + \eta_1(c)|d|^\nu, \quad \forall (c,d)\in\mathcal{S}, \tag{3.4}$$

for some nonnegative locally bounded real functions γ_1, η_1. Moreover, due Assumption 2.2, the function $\hat{V}$ is continuous in $\mathcal{S}$ and differentiable with respect to c for all $d \in \mathcal{O}$, with

$$\hat{V}_c(c,d) \;=\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t} g_c(c, D_t^d)\mathrm{d}t\Big], \quad (c,d)\in\mathcal{S}, \tag{3.5}$$

and for the same reason as before

$$\hat{V}_c(c,d) \;\leq\; \gamma_1(c) + \eta_1(c)|d|^\nu, \quad \forall (c,d)\in\mathcal{S}. \tag{3.6}$$

Now, let $d_0 \in \mathcal{O}$ be a reference point and let us introduce the functions

$$S'(d) \;:=\; \exp\left(-\int_{d_0}^d \frac{2\mu(\xi)\mathrm{d}\xi}{\sigma^2(\xi)}\right), \quad d\in\mathcal{O},$$

and

$$m'(d) \;:=\; \frac{2}{\sigma^2(d)S'(d)}, \quad d\in\mathcal{O}.$$

S' is the the density of the so called scale function of the diffusion D, and m' is the density of the so called speed measure of the diffusion D. Let us denote respectively by ψ and φ the increasing and decreasing fundamental solutions, individuated up to a multiplicative constant, to the linear ordinary differential equation

$$\mathcal{L}\phi(d) \;:=\; \rho\phi(d) - \mu(d)\phi'(d) - \frac{1}{2}\sigma^2(d)\phi''(d) \;=\; 0. \tag{3.7}$$

The existence and properties of such functions, as well as their relationship with the functions S, m defined above, can be found in several references including in [8, Ch. II], [29, Ch. 15], [39, Ch. V], and [32, Ch. 2]. In particular we know that ψ, φ are strictly positive, convex, and, since $d_{\min}, d_{\max}$ are natural boundaries, they satisfy (see, e.g., [8, Ch. 2])

$$\lim_{d \downarrow d_{\min}} \psi(d) \;=\; 0, \quad \lim_{d \downarrow d_{\min}} \varphi(d) \;=\; \infty, \quad \lim_{d \uparrow d_{\max}} \psi(d) \;=\; \infty, \quad \lim_{d \uparrow d_{\max}} \varphi(d) \;=\; 0, \tag{3.8}$$

$$\lim_{d \downarrow d_{\min}} \frac{\psi'(d)}{S'(d)} \;=\; 0, \quad \lim_{d \downarrow d_{\min}} \frac{\varphi'(d)}{S'(d)} \;=\; -\infty, \quad \lim_{d \uparrow d_{\max}} \frac{\psi'(d)}{S'(d)} \;=\; \infty, \quad \lim_{d \uparrow d_{\max}} \frac{\varphi'(d)}{S'(d)} \;=\; 0. \tag{3.9}$$

Let w be the constant positive Wronskian of the fundamental solutions ψ, φ, i.e.

$$0 \;<\; w \;\equiv\; \frac{\psi'(d)\varphi(d) - \psi(d)\varphi'(d)}{S'(d)}, \qquad d \in \mathcal{O}.$$

and let $p(t, d, \cdot)$ be the density of the transition probability $P(t, d, \cdot)$ of the diffusion D. Using the characterization of the Green's function

$$G(d, h) \;:=\; \int_0^\infty e^{-\rho t} p(t, d, h) \mathrm{d}t$$

associated to D as

$$G(d, h) \;=\; \begin{cases} w^{-1}\psi(d)\varphi(h), & \text{if } d \leq h, \\ w^{-1}\psi(h)\varphi(d), & \text{if } d \geq h, \end{cases}$$

and the fact that it is the kernel of the resolvent operator (see, e.g., [39, Ch. V] or [29, Ch. 15]) with respect to m, i.e.

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} f(D_t^d) \mathrm{d}t\right] \;=\; \int_{\mathcal{O}} f(h) G(d, h) m'(h) \mathrm{d}h, \qquad \forall f \in \mathcal{B}(\mathcal{O}; \mathbb{R}),$$

we see (approximating g, g_c by bounded functions and using the monotone convergence theorem) that the functions $\hat{V}$ and $\hat{V}_c$ can be represented in terms of ψ, φ as

$$\hat{V}(c, d) \;=\; w^{-1}\Big[\varphi(d) \int_{d_{\min}}^{d} \psi(\xi) g(c, \xi) m'(\xi) \mathrm{d}\xi \;+\; \psi(d) \int_{d}^{d_{\max}} \varphi(\xi) g(c, \xi) m'(\xi) \mathrm{d}\xi\Big], \tag{3.10}$$

$$\hat{V}_c(c, d) \;=\; w^{-1}\Big[\varphi(d) \int_{d_{\min}}^{d} \psi(\xi) g_c(c, \xi) m'(\xi) \mathrm{d}\xi \;+\; \psi(d) \int_{d}^{d_{\max}} \varphi(\xi) g_c(c, \xi) m'(\xi) \mathrm{d}\xi\Big]. \tag{3.11}$$

Proposition 3.1. *The value function v is convex with respect to c and satisfies the growth condition, for some locally bounded functions $\gamma_1, \eta_1 : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$0 \;\leq\; v(c, d) \;\leq\; \hat{V}(c, d) \;\leq\; \gamma_1(c) + \eta_1(c)|d|^\nu, \quad \forall (c, d) \in \mathcal{S}, \tag{3.12}$$

Proof. (3.12) comes from (3.5) and from the inequality $v(c, d) \leq G(c, d; 0) = \hat{V}(c, d)$.

Convexity of v follows in a standard way from the convexity of g with respect to c and linearity of the state equation for $C^{c,I}$. □

3.2 Existence of optimal controls and the associated Dynkin game

In this subsection we show that the singular stochastic control problem admits optimal controls and that it is related to a suitable associated Dynkin game. We estabilish this connection mainly to inherit from the monotonicity of $g_c(c,\cdot)$ the monotonicity of $v_c(c,\cdot)$, whose direct proof seems not attainable. The proofs of Propositions 3.2, 3.3 closely follow the arguments of [28], and are reported in Appendix.

Definition 3.1. *Given $(c,d)\in\mathcal{S}$ we say that a control $I^*\in\mathcal{I}$ is optimal starting from (c,d) if $G(c,d;I^*)=v(c,d)$.*

Proposition 3.2. *For all $(c,d)\in\mathcal{S}$ there exists an optimal control I^* starting from (c,d). Moreover, if $g(\cdot,d)$ is strictly convex on $\mathbb{R}$ for every $d\in\mathcal{O}$, then I^* is the unique (up to undistinguishability) optimal control starting from (c,d).*

Let $\mathcal{T}$ denote the set of all $\mathbb{F}$-stopping times. For fixed $(c,d)\in\mathcal{S}$, we may consider the functional, controlled by $\sigma\in\mathcal{T},\ \tau\in\mathcal{T}$,

$$J(c,d;\sigma,\tau) \;=\; \mathbb{E}\Big[\int_0^{\sigma\wedge\tau} e^{-\rho t}g_c(c,D_t^d)\mathrm{d}t+q_0^-e^{-\rho\sigma}\mathbf{1}_{\{\sigma<\tau\}}-q_0^+e^{-\rho\tau}\mathbf{1}_{\{\tau<\sigma\}}\Big]. \tag{3.13}$$

We can imagine that $J(c,d;\tau,\sigma)$ is the payoff associated to a two-players stochastic game. The two players, P1 and P2, have the possibility to stop the game at times σ and τ, respectively (i.e. P1 controls the game through σ and P2 controls the game through τ). If P1 stops first $(\sigma<\tau)$, he pays to P2 the amount $q_0^-e^{-\rho\sigma}$; if P2 stops first $(\tau<\sigma)$, he pays to P1 the amount $q_0^+e^{-\rho\tau}$; if they decide to stop at the same time, i.e. if $\tau=\sigma$, then no cashflow occurs; finally, as long as the game is running, i.e. up to time $\sigma\wedge\tau$, P1 pays P2 at the rate $e^{-\rho t}g_c(c,D_t^d)$ per unit of time. The goal of P1 is to minimize (3.13), while the goal of P2 is to maximize (3.13). The functions

$$\underline{w}(c,d) \;:=\; \sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}} J(c,d;\sigma,\tau), \qquad \overline{w}(c,d) \;:=\; \inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}} J(c,d;\sigma,\tau),$$

are called *lower- and upper-values of the game.* Clearly $\underline{w}(c,d)\le\overline{w}(c,d)$. If $\underline{w}(c,d)=\overline{w}(c,d)$, the game is said to have a *value* denoted by $w(c,d):=\underline{w}(c,d)=\overline{w}(c,d)$. A pair $(\sigma^*,\tau^*)\in\mathcal{T}\times\mathcal{T}$ is called a *saddle-point of the game* if

$$J(c,d;\sigma^*,\tau) \;\le\; J(c,d;\sigma^*,\tau^*) \;\le\; J(c,d;\sigma,\tau^*), \qquad \forall\sigma\in\mathcal{T},\ \ \forall\tau\in\mathcal{T}. \tag{3.14}$$

One easily sees that the existence of a saddle point implies that the game has a value and

$$w(c,d) \;=\; J(c,d;\sigma^*,\tau^*). \tag{3.15}$$

Proposition 3.3. *1. Let $(c,d)\in\mathcal{S}$ and let $I^*=(I^{*,+},I^{*,-})\in\mathcal{I}$ be an optimal control for the singular stochastic control problem, i.e. such that $v(c,d)=G(c,d;I^*)$. Define the stopping times*

$$\sigma^* \;:=\; \inf\{\,t\ge 0\ \mid\ I_t^{*,-}>0\,\}, \qquad \tau^* \;:=\; \inf\{\,t\ge 0\ \mid\ I_t^{*,+}>0\,\}.$$

Then $(\sigma^*,\tau^*)\in\mathcal{T}\times\mathcal{T}$ is a saddle point for the associated Dynkin game.

2. *v is differentiable with respect to c in $\mathcal{S}$ and it holds the equality $v_c = w$, where w is the (well-defined) value of the associated Dynkin game.*

By relying on this connection between singular control and Dynkin game, we prove now some properties on the derivative of the value function v_c, to be used in the next Section.

Proposition 3.4. *The function v_c has the following properties:*

1. *v_c is continuous in $\mathcal{S}$.*
2. *$v_c(c, \cdot)$ is nonincreasing in $\mathcal{O}$ for all $c \in \mathbb{R}$.*
3. *$-q_0^+ \leq v_c \leq q_0^-$ in $\mathcal{S}$.*

Proof. 1. Let $(c, d) \in \mathcal{S}$ and take a sequence $(c_n, d_n) \to (c, d)$. For each $n \in \mathbb{N}$, let (σ_n^*, τ_n^*) be a saddle-point for the Dynkin game starting at (c_n, d_n), and let (σ^*, τ^*) be a saddle point for the Dynkin game starting at (c, d). Using (3.14), we then have

$$\begin{aligned} w(c,d) - w(c_n, d_n) &= J(c, d; \sigma^*, \tau^*) - J(c_n, d_n; \sigma_n^*, \tau_n^*) \\ &\leq J(c, d; \sigma_n^*, \tau^*) - J(c_n, d_n; \sigma_n^*, \tau^*) \\ &= \mathbb{E}\left[\int_0^{\tau^* \wedge \sigma_n^*} e^{-\rho t} \big(g_c(c, D_t^d) - g_c(c_n, D_t^{d_n})\big) \mathrm{d}t\right] \qquad (3.16) \\ &= \mathbb{E}\left[\int_0^{\infty} e^{-\rho t} \big(g_c(c, D_t^d) - g_c(c_n, D_t^{d_n})\big) \mathbf{1}_{\{t \leq \tau^* \wedge \sigma_n^*\}} \mathrm{d}t\right]. \end{aligned}$$

Note that, assuming without loss of generality that $(d_n)_{n \in \mathbb{N}} \subset (d - \varepsilon, d - \varepsilon) \subset \mathcal{O}$ for suitable $\varepsilon > 0$, we have by Assumption 2.1 (iv)

$$|D_t^{d_n}| \leq |D_t^{d-\varepsilon}| + |D_t^{d+\varepsilon}|, \quad \forall t \geq 0, \ \forall n \in \mathbb{N}. \qquad (3.17)$$

On the other hand Assumption 2.1 (iii) ensures the convergence

$$D_t^{d_n} \stackrel{n \to \infty}{\longrightarrow} D_t^d, \quad \text{a.s.}, \quad \forall t \geq 0. \qquad (3.18)$$

Hence, using Assumption 2.2, (3.3), and (3.17)-(3.18), we can apply dominated convergence to (3.16) for $n \to \infty$ and conclude that $\liminf_{n \to \infty} w(c_n, d_n) \geq w(c, d)$.

Arguing in a similar way, but considering the couple (σ^*, τ_n^*) in place of the couple (σ_n^*, τ^*), one also gets the inequality $\limsup_{n \to \infty} w(c_n, d_n) \leq w(c, d)$, so w is continuous at (c, d).

Then the claim follows by Proposition 3.3 (2).

2. By the assumption that $g_c(c, \cdot)$ is non increasing (Assumption 2.2(ii)), and from the same comparison result cited above, we have, for every $d, d' \in \mathcal{O}$ such that $d \leq d'$,

$$J(c, d; \sigma, \tau) \geq J(c, d'; \sigma, \tau), \qquad \forall \sigma \in \mathcal{T}, \ \forall \tau \in \mathcal{T}.$$

Passing to the infimum over $\sigma \in \mathcal{T}$ and then to the supremum over $\tau \in \mathcal{T}$ the inequality above we get, for every $d, d' \in \mathcal{O}$ such that $d \leq d'$,

$$\underline{w}(c, d) \geq \underline{w}(c, d').$$

Proposition 3.3 states that the game has a value, so from the inequality above we get, for every $d, d' \in \mathcal{O}$ such that $d \leq d'$,

$$w(c,d) \quad \geq \quad w(c,d').$$

Hence, the claim follows from Proposition 3.3,(2).

3. We have $J(c,d;\sigma,0) = -q_0^+$ for every $\sigma \in \mathcal{T}$, and $J(c,d;0,\tau) = q_0^-$ for every $\tau \in \mathcal{T}$. It follows that $-q_0^+ \leq w(c,d) \leq q_0^-$ and the claim follows from Proposition 3.3 (2). □

4 The dynamic programming equation and the structure of the solution

In view of Proposition 3.4, we introduce the so-called *continuation region*

$$\mathcal{C} \quad := \quad \{\, (c,d) \in \mathcal{S} \mid -q_0^+ < v_c(c,d) < q_0^- \,\},$$

and its complement set, the *action region*

$$\mathcal{A} \quad := \quad \mathcal{A}^+ \,\cup\, \mathcal{A}^-, \tag{4.1}$$

where $\mathcal{A}^+$ and $\mathcal{A}^-$ are respectively the *investment* and the *disinvestment region* defined by

$$\mathcal{A}^+ \;:=\; \{(c,d) \in \mathcal{S} \mid v_c(c,d) = -q_0^+\}, \quad \mathcal{A}^- \;:=\; \{(c,d) \in \mathcal{S} \mid v_c(c,d) = q_0^-\}. \tag{4.2}$$

We also set

$$\partial^+\mathcal{C} \;=\; \bar{\mathcal{C}} \cap \mathcal{A}^+, \qquad \partial^-\mathcal{C} \;=\; \bar{\mathcal{C}} \cap \mathcal{A}^-.$$

The boundaries $\partial^\pm\mathcal{C}$ are associated with a free boundary differential problem (which we are going to define in the next subsection) and are the objects to individuate to solve the optimal stochastic control problem.

Let us then consider the functions $\hat{c}_+, \hat{c}_- : \mathcal{O} \to \bar{\mathbb{R}}$ defined with the conventions $\inf \emptyset = \infty$, $\inf \mathbb{R} = -\infty$, $\sup \mathbb{R} = \infty$, $\sup \emptyset = -\infty$ (the equalities below are consequence of convexity of v with respect to c):

$$\hat{c}_+(d) \;:=\; \inf\,\{c \in \mathbb{R} \mid v_c(c,d) > -q_0^+\} \;=\; \sup\,\{c \in \mathbb{R} \mid v_c(c,d) = -q_0^+\}, \tag{4.3}$$

$$\hat{c}_-(d) \;:=\; \sup\,\{c \in \mathbb{R} \mid v_c(c,d) < q_0^-\} \;=\; \inf\,\{c \in \mathbb{R} \mid v_c(c,d) = q_0^-\}. \tag{4.4}$$

Proposition 4.1. *1. $\hat{c}_+ : \mathcal{O} \to \mathbb{R} \cup \{-\infty\}$, $\hat{c}_- : \mathcal{O} \to \mathbb{R} \cup \{\infty\}$, they are both nondecreasing and*

$$\hat{c}_+(d) \;<\; \hat{c}_-(d), \qquad \forall d \in \mathcal{O}. \tag{4.5}$$

2. $\hat{c}_+$ is right-continuous and $\hat{c}_-$ is left-continuous.

3. The action and continuation regions are expressed in terms of the functions $\hat{c}_\pm$ as:

$$\mathcal{C} \;=\; \{(c,d) \in \mathcal{S} \mid \hat{c}_+(d) < c < \hat{c}_-(d)\},$$

$$\mathcal{A}^+ \;=\; \{(c,d) \in \mathcal{S} \mid c \leq \hat{c}_+(d)\}, \qquad \mathcal{A}^- \;=\; \{(c,d) \in \mathcal{S} \mid c \geq \hat{c}_-(d)\}.$$

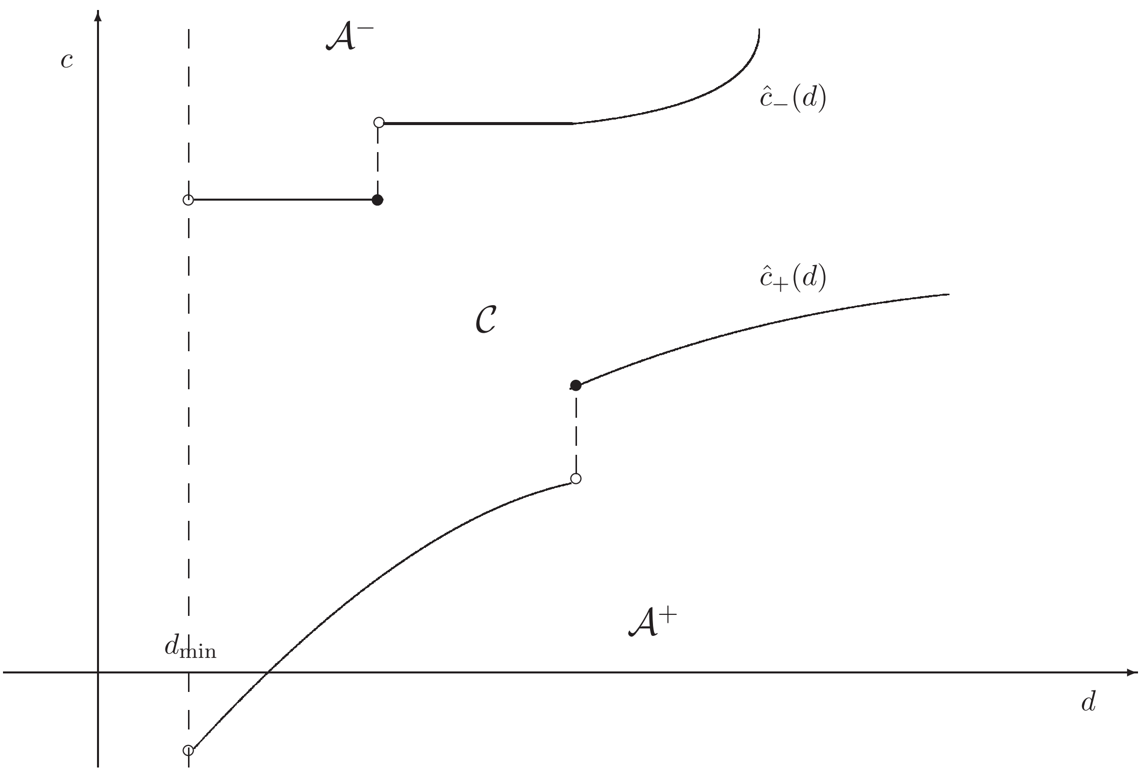
4. $\mathcal{C}$ is open and connected, and $\mathcal{A}^{\pm}$ are closed and connected.

Proof. 1. The fact that $\hat{c}_+$ takes values in $\mathbb{R} \cup \{-\infty\}$ and $\hat{c}_-$ takes values in $\mathbb{R} \cup \{\infty\}$ is consequence of the nonnegativity of v, combined with the convexity of $v(\cdot, d)$ and with (4.3)-(4.4). Monotonicity follows from Proposition 3.4 (2) and (4.3)-(4.4). Finally, (4.5) is due to the convexity of v with respect to c and to the fact that $v(\cdot, d) \in C^1(\mathbb{R};\mathbb{R})$ for every $d \in \mathcal{O}$.

2. It follows from Proposition 3.4 (1) and from the convexity of v w.r.t. c.

3-4. They follow from the previous items also considering (4.3)-(4.4). □

Below it is represented a possible shape of the regions $\mathcal{C}, \mathcal{A}^{\pm}$ and of the functions $\hat{c}^{\pm}$ (here $d_{\max} = \infty$).



Let us define

$$\underline{c}_+ := \inf_{d \in \mathcal{O}} \hat{c}_+(d), \quad \bar{c}_+ := \sup_{d \in \mathcal{O}} \hat{c}_+(d), \quad \underline{c}_- := \inf_{d \in \mathcal{O}} \hat{c}_-(d), \quad \bar{c}_- := \sup_{d \in \mathcal{O}} \hat{c}_-(d),$$

and the pseudo-inverse of $\hat{c}_\pm$, i.e. the functions $\hat{d}_\pm : \mathbb{R} \longrightarrow \bar{\mathcal{O}}$ defined by

$$\hat{d}_+(c) := \inf \{d \in \mathcal{O} \mid \hat{c}_+(d) \geq c\}, \quad \hat{d}_-(c) := \sup \{d \in \mathcal{O} \mid \hat{c}_-(d) \leq c\}, \tag{4.6}$$

with the convention $\inf \emptyset = d_{\max}$ and $\sup \emptyset = d_{\min}$.

Proposition 4.2. *1. We have the equalities*

$$\hat{d}_+(c) \;=\; \sup\,\{d \in \mathcal{O} \mid v_c(c,d) > -q_0^+\}, \qquad \hat{d}_-(c) \;=\; \inf\,\{d \in \mathcal{O} \mid v_c(c,d) < q_0^-\}. \quad (4.7)$$

2. The functions $\hat{d}_\pm$ are nondecreasing and $\hat{d}_+ \geq \hat{d}_-$.

3. If $\bar{c}_- < \infty$, then $\hat{d}_- = d_{\max}$ on $[\bar{c}_-, \infty)$; if $\underline{c}_+ > -\infty$, then $\hat{d}_+ = d_{\min}$ on $(-\infty, \underline{c}_+]$.

4. $\hat{d}_-(c) < \hat{d}_+(c)$ if and only if $c \in (\underline{c}_+, \bar{c}_-)$.

Proof. 1. It directly follows from the definition of $\hat{c}_\pm$, $\hat{d}_\pm$.

2. Monotonicity of $\hat{d}_\pm$ and the inequality $\hat{d}_+ \geq \hat{d}_-$ follow from Proposition 4.1 (1).

3. By monotonicity of $\hat{d}_-$, $\lim_{c\to\infty+} \hat{d}_-(c)$ exists. Suppose by contradiction $\lim_{c\to\infty} \hat{d}_-(c) = \bar{d} < d_{\max}$. This would imply $\hat{c}_- = \infty$ over $(\bar{d}, d_{\max})$, which contradicts $\bar{c}_- < \infty$. A similar argument works for the other claim.

4. It follows from (4.5). □

We also introduce the c-section sets of the continuation region

$$S_c \;:=\; \{c\} \times (\hat{d}_-(c), \hat{d}_+(c)), \quad c \in \mathbb{R}. \quad (4.8)$$

Due to Proposition 4.2, we have

$$c \in (\underline{c}_+, \bar{c}_-) \iff \hat{d}_-(c) < \hat{d}_+(c) \iff S_c \neq \emptyset. \quad (4.9)$$

We have the following result on the form of the continuation region.

Proposition 4.3. *We have the representation of the continuation region*

$$\mathcal{C} \;=\; \bigcup_{c \in (\underline{c}_+, \bar{c}_-)} S_c. \quad (4.10)$$

Proof. If $(c,d) \in \mathcal{C}$, then $-q_0^+ < v_c(c,d) < q_0^-$, so, by continuity of v_c (Proposition 3.4 (1)), it is $-q_0^+ < \hat{v}_c < q_0^-$ in some suitable neighborhood of (c,d). Then $\hat{d}_-(c) < \hat{d}_+(c)$, therefore, by (4.9), $c \in (\underline{c}_+, \bar{c}_-)$ and $(c,d) \in S_c \neq \emptyset$. Hence we have proved the inclusion $\mathcal{C} \subset \bigcup_{c\in(\underline{c}_+,\bar{c}_-)} S_c$.

Conversely, let $c \in (\underline{c}_+, \bar{c}_-)$ and let $d \in \mathcal{O}$ be such that $(c,d) \in S_c(\neq \emptyset)$. By (4.7) and (4.9), we have $-q_0^+ < v_c(c,\cdot) < q_0^-$ in some neighborhood of d. The continuity of v_c with respect to c (Proposition 3.4 (1)) implies $-q_0^+ < v_c < q_0^-$ in some neighborhood of (c,d). Therefore $(c,d) \in \mathcal{C}$. Hence we have proved the inclusion $\mathcal{C} \supset \bigcup_{c\in(\underline{c}_+,\bar{c}_-)} S_c$. □

We also introduce the functions $\hat{c}_{\pm,g}$ from $\mathcal{O}$ into $\overline{\mathbb{R}}$ defined, with the usual convention $\sup \emptyset = -\infty, \inf \emptyset = \infty$, by:

$$\hat{c}_{+,g}(d) \;=\; \inf\{c \in \mathbb{R} \mid g_c(c,d) > -\rho q_0^+\}, \qquad \hat{c}_{-,g}(d) \;=\; \sup\{c \in \mathbb{R} \mid g_c(c,d) < \rho q_0^-\}.$$

One easily checks that, by Assumption 2.2, they are nondecreasing and, respectively right- and left-continuous. Moreover, we clearly have, by convexity of $g(\cdot,d)$ and continuity of g_c, the inequality $\hat{c}_{+,g} < \hat{c}_{-,g}$. We have the following estimates of $\hat{c}_\pm$ in terms of $\hat{c}_{\pm,g}$.

Proposition 4.4. *$\hat{c}_+ \le \hat{c}_{+,g}$ and $\hat{c}_- \ge \hat{c}_{-,g}$.*

Proof. Let us show the first inequality, the second one can be proved symmetrically. Let $d \in \mathcal{O}$ and take $c > \hat{c}_{+,g}(d)$, so that $g_c(c,d) + \rho q_0^+ > 0$. Let $\varepsilon \in \Big(0, \frac{g_c(c,d)+\rho q_0^+}{\rho}\Big)$, and consider the stopping time

$$\tau_\varepsilon \quad := \quad \inf\{t \ge 0 \mid g_c(c, D_t^d) + \rho q_0^+ \le \rho\varepsilon\}.$$

By continuity of $g_c(c,\cdot)$ and by continuity of trajectories of D^d, we have $\tau_\varepsilon > 0$. Then, by Proposition 3.3 (2) and taking into account the definition of τ_ε, we have

$$\begin{aligned} v_c(c,d) \quad &= \quad \inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} J(c,d;\sigma,\tau) \quad \ge \quad \inf_{\sigma\in\mathcal{T}} J(c,d;\sigma,\tau_\varepsilon) \\ &= \quad \inf_{\sigma\in\mathcal{T}} \mathbb{E}\Big[\int_0^{\sigma\wedge\tau_\varepsilon} e^{-\rho t} g_c(c, D_t^d)\mathrm{d}t + q_0^- e^{-\rho\sigma}\mathbf{1}_{\{\sigma<\tau_\varepsilon\}} - q_0^+ e^{-\rho\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\sigma\}}\Big] \\ &\ge \quad \inf_{\sigma\in\mathcal{T}} \mathbb{E}\Big[(\varepsilon - q_0^+)(1 - e^{-\rho(\sigma\wedge\tau_\varepsilon)}) + q_0^- e^{-\rho\sigma}\mathbf{1}_{\{\sigma<\tau_\varepsilon\}} - q_0^+ e^{-\rho\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\sigma\}}\Big] \\ &\ge \quad \inf_{\sigma\in\mathcal{T}} \mathbb{E}\Big[\varepsilon(1 - e^{-\rho\tau_\varepsilon})\mathbf{1}_{\{\tau_\varepsilon<\sigma\}} - q_0^+ e^{-\rho\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\sigma\}}\Big]. \end{aligned}$$

Clearly the last term of the inequality above is larger than $-q_0^+$. Now, assume by contradiction that it is equal to $-q_0^+$. This means that there exists a minimizing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\varepsilon(1 - e^{-\rho\tau_\varepsilon})\mathbf{1}_{\{\tau_\varepsilon<\sigma_n\}} - q_0^+ e^{-\rho\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\sigma_n\}}\Big] \quad = \quad -q_0^+. \tag{4.11}$$

Hence, looking at the second addend in the expectation above, since the first one is nonnegative, we see that we must have $\mathbb{P}\{\tau_\varepsilon < \sigma_n\} \to 1$. But then we must have

$$(1 - e^{\rho\tau_\varepsilon})\mathbf{1}_{\{\tau_\varepsilon<\sigma_n\}} \quad \xrightarrow{\mathbb{P}} \quad 1 - e^{-\rho\tau_\varepsilon} > 0,$$

from which we deduce that

$$\lim_{n\to\infty} \mathbb{E}\Big[\varepsilon(1 - e^{-\rho\tau_\varepsilon})\mathbf{1}_{\{\tau_\varepsilon<\sigma_n\}}\Big] \quad > \quad 0,$$

contradicting (4.11). So we have shown that $v_c(c,d) > -q_0^+$. By continuity of $v_c(c,\cdot)$, this shows that $c > \hat{c}_+(d)$, completing the proof. □

4.1 The dynamic programming equation

The dynamic programming equation for the singular stochastic control problem (3.1) takes the form of a variational inequality:

$$\max\Big\{ [\mathcal{L}v(c,\cdot)](d) - g(c,d),\ -v_c(c,d) - q_0^+,\ v_c(c,d) - q_0^- \Big\} \quad = \quad 0, \quad (c,d) \in \mathcal{S}, \tag{4.12}$$

where the second-order ordinary differential operator $\mathcal{L}$ is defined in (3.7). Formally, (4.12) may be derived, assuming sufficient regularity of v and exploiting its convexity in c, by looking at the three possibilities one has: (1) wait; (2) invest a small amount ε; (3) disinvest a small amount ε. We refer to [17] for a formal derivation of the dynamic programming equation in the general context of singular control problems, and specifically to [34] for a problem very similar to ours.

In the following, given a locally bounded function $\phi : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U} \subset \mathbb{R}^n$ is an open set, we denote respectively by ϕ^*, and ϕ_* the upper semicontinuous and the lower semicontinuous envelope of ϕ. Since we do not know a priori if there exists a smooth solution to (4.12), we first rely in general on the notion of viscosity solutions:

Definition 4.1. *(i) We say that v is a viscosity subsolution to* (4.12) *if for any $(c,d) \in \mathcal{S}$,*

$$\max\left\{ [\mathcal{L}\varphi(c,\cdot)](d) - g(c,d),\ -\varphi_c(c,d) - q_0^+,\ \varphi_c(c,d) - q_0^- \right\} \ \leq \ 0,$$

whenever $\varphi \in C^{1,2}(\mathcal{S};\mathbb{R})$, $v^*(c,d) = \varphi(c,d)$, and $v^* - \varphi$ has a local maximum at (c,d).
(ii) We say that v is a viscosity supersolution to (4.12) *if for any $(c,d) \in \mathcal{S}$,*

$$\max\left\{ [\mathcal{L}\varphi(c,\cdot)](d) - g(c,d),\ -\varphi_c(c,d) - q_0^+,\ \varphi_c(c,d) - q_0^- \right\} \ \geq \ 0,$$

whenever $\varphi \in C^{1,2}(\mathcal{S};\mathbb{R})$, $v_*(c,d) = \varphi(c,d)$, and $v_* - \varphi$ has a local minimum at (c,d).
(iii) We say that v is a viscosity solution to (4.12) *if it is both a viscosity sub- and supersolution.*

The viscosity property of the value function follows usually from the dynamic programming principle (DPP). The statement of DPP calls upon delicate measurable selection arguments. Once we know a priori that the value function is continuous, one can overcome this difficulty by exploiting the continuity, see e.g. [17]. However, since the control set is unbounded, and we are not assuming Lipschitz continuity of the coefficients in (2.1) and - overall - of g, it is not clear how to get the continuity of the value function from its very definition. Instead, we can use the concept of weak dynamic programming introduced in [9], which holds for our problem (see also Remarks 3.10 and 3.11 in [9]), stating that, for each $(c,d) \in \mathcal{S}$ and for each family $(\tau_I)_{I\in\mathcal{I}}$ of stopping times indexed by $I \in \mathcal{I}$, it holds

$$\begin{aligned}
&\inf_{I\in\mathcal{I}} \ \mathbb{E}\left[\int_0^{\tau_I^-} e^{-\rho t} g(C_t^{c,I}, D_t^d)\mathrm{d}t + q_0^+ \mathrm{d}I_t^+ + q_0^- \mathrm{d}I_t^- + e^{-\rho\tau_I} v_*(C_{\tau_I^-}^{c,I}, D_{\tau_I}^d)\right] \\
&\leq \ v(c,d) \ \leq \ \inf_{I\in\mathcal{I}} \ \mathbb{E}\left[\int_0^{\tau_I^-} e^{-\rho t} g(C_t^{c,I}, D_t^d)\mathrm{d}t + q_0^+ \mathrm{d}I_t^+ + q_0^- \mathrm{d}I_t^- + e^{-\rho\tau_I} v^*(C_{\tau_I^-}^{c,I}, D_{\tau_I}^d)\right]. \qquad (4.13)
\end{aligned}$$

Proposition 4.5. *The value function v is a viscosity solution to* (4.12) *on $\mathcal{S}$.*

Proof. Given the weak DPP (4.13), the proof is straightforward (and we omit it for brevity), and follows the line of the proof based on the standard Dynamic Programming Principle. Indeed, what one really needs are the two inequalities of (4.13) separately to prove the two viscosity properties separately. We can refer to [9, Sec. 5] where this is done for the case of continuous control; the proof can be adapted to our case of stochastic control. □

Remark 4.1. A comparison principle to the variational inequality (4.12) for viscosity sub-and super solution satisfying the growth condition (3.12) could be proved using standard techniques (see [14]), hence providing a uniqueness viscosity characterization of the value function v. However, in our approach we rely mainly on the viscosity property in order to derive a smooth-fit property. □

We now investigate the structure of the value function v in the continuation region $\mathcal{C}$ and in the action regions $\mathcal{A}^{\pm}$. The following lemma characterizes the structure of v in the c-sections S_c defined in (4.8).

Lemma 4.1. *Let $c \in (\underline{c}_+, \bar{c}_-)$.*

1. *$v(c,\cdot)$ is a viscosity solution of the ODE*

$$[\mathcal{L}v(c,\cdot)](d) - g(c,d) \;=\; 0, \qquad d \in (\hat{d}_+(c), \hat{d}_-(c)). \tag{4.14}$$

2. *$v(c,\cdot) \in C^2((\hat{d}_-(c), \hat{d}_+(c)); \mathbb{R})$.*

3. *There exist constants $A(c), B(c) \in \mathbb{R}$ such that*

$$v(c,d) \;=\; A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c,d), \quad \forall\, d \in (\hat{d}_-(c), \hat{d}(c)). \tag{4.15}$$

Moreover, (4.15) holds also at $\hat{d}_-(c), \hat{d}_+(c)$ when they do not coincide with $d_{\min}, d_{\max}$, respectively.

Proof. 1. Let us show the subsolution property (the proof of the supersolution property is completely analogous).

First of all we note that, since $v(\cdot, d) \in C^1(\mathbb{R};\mathbb{R})$, it is $v(c,d) = v(c_0,d) + \int_{c_0}^{c} v_c(\xi,d)\mathrm{d}\xi$, for every $c, c_0 \in \mathbb{R}$ and every $d \in \mathcal{O}$. Thus, since by Proposition 3.4 (1) v_c is continuous in $\mathcal{S}$, we deduce the equalities

$$\begin{aligned} v^*(c,d) &= v(c,\cdot)^*(d), && \forall (c,d) \in \mathcal{S}; && (4.16)\\ v^*(c,d) - v^*(c_0,d) &= v(c,d) - v(c_0,d), && \forall (c,d) \in \mathcal{S},\ \forall c_0 \in \mathbb{R}. && (4.17) \end{aligned}$$

Let $c_0 \in (\underline{c}_+, \bar{c}_-)$, $d_0 \in (\hat{d}_+(c_0), \hat{d}_-(c_0))$, and let $\phi \in C^2(\mathcal{O};\mathbb{R})$ be such that

$$\phi(d_0) \;=\; v(c_0,\cdot)^*(d_0), \qquad \phi(d) \;\geq\; v(c,\cdot)^*(d), \quad \forall d \in \mathcal{O}. \tag{4.18}$$

We claim that

$$(v_c(c_0,d_0), \phi'(d_0), \phi''(d_0)) \;\in\; D^{1,2,+}_{c,d} v^*(c_0,d_0), \tag{4.19}$$

where $D^{1,2,+}_{c,d} v^*(c_0,d_0)$ is the superdifferential of v^* at (c_0,d_0) of first order w.r.t. c and of second order w.r.t. d (see [43], Ch. 4, Sec. 5). We have to check that

$$\limsup_{(c,d)\to(c_0,d_0)} \frac{v^*(c,d) - v^*(c_0,d_0) - v_c(c_0,d_0)(c-c_0) - \phi'(d_0)(d-d_0) - \phi''(d_0)(d-d_0)^2}{|c-c_0| + |d-d_0|^2} \;\leq\; 0. \tag{4.20}$$

By (4.16) it has to be $(\phi'(d_0), \phi''(d_0)) \in D^{2,+}_{d} v^*(c_0,d_0)$, where $D^{2,+}_{d} v^*(c_0,d_0)$ is the superdifferential of v^* at (c_0,d_0) of second order w.r.t. d. Hence

$$v^*(c_0,d) - v^*(c_0,d_0) - \phi'(d_0)(d-d_0) - \phi''(d_0)(d-d_0)^2 \;\leq\; o(|d-d_0|^2). \tag{4.21}$$

Moreover, since $v(\cdot,d) \in C^1(\mathbb{R};\mathbb{R})$ for every $d \in \mathcal{O}$ and v_c is locally uniformly continuous w.r.t. $(c,d) \in \mathcal{S}$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} v(c,d) - v(c_0,d) - v_c(c_0,d)(c-c_0) &\leq o(|c-c_0|), \text{ unif. in } d \in (d_0-\delta, d_0+\delta), && (4.22)\\ |v_c(c_0,d) - v_c(c_0,d_0)| &\leq \varepsilon, \qquad \forall\, d \in (d_0-\delta, d_0+\delta). && (4.23) \end{aligned}$$

By (4.17), we derive from (4.22)

$$v^*(c,d) - v^*(c_0,d) - v_c(c_0,d)(c-c_0) \quad \leq \quad o(|c-c_0|), \text{ unif. in } d \in (d_0-\delta, d_0+\delta). \quad (4.24)$$

By subtracting and adding $v_c(c_0,d_0)(c-c_0)$ in (4.24) and using (4.23), we get

$$v^*(c,d) - v^*(c_0,d) - v_c(c_0,d_0)(c-c_0) \leq o(|c-c_0|) + \varepsilon \cdot |c-c_0|, \text{ unif. in } d \in (d_0-\delta, d_0+\delta). (4.25)$$

Combining (4.21) and (4.25), dividing by $|c-c_0| + |d-d_0|^2$, and taking the limsup, since ε was arbitrary, we finally get (4.20), thus (4.19).

Now, starting from (4.19), we can construct (see, e.g., [43], Ch. 4, Lemma 5.4[4]) a function $\varphi \in C^{1,2}(\mathcal{S};\mathbb{R})$ such that $\varphi(c_0,d_0) = v^*(c_0,d_0)$, $\varphi \geq v^*$ on $\mathcal{S}$ and

$$(\varphi_c(c_0,d_0), \varphi_d(c_0,d_0), \varphi_{dd}(c_0,d_0)) \quad = \quad (v_c(c_0,d_0), \phi'(d_0), \phi''(d_0)). \quad (4.26)$$

Now notice that $-q_0^+ < v_c(c_0,d_0) < q_0^-$, as $(c_0,d_0) \in \mathcal{C}$ (Proposition 4.3). Hence, since v is a viscosity solution to (4.12), taking into account (4.26) we finally get the desired inequality $[\mathcal{L}\phi](d_0) \leq 0$.

2. Let $c \in (\underline{c}_+, \bar{c}_-)$ and, given $a, b \in \bar{S}_c$ with $a < b$, consider the Dirichlet problem

$$\begin{cases} \rho u(d) - \mu(d)u'(d) - \frac{1}{2}\sigma^2(d)u''(d) \ = \ g(c,d), \quad d \in (a,b), \\ u(a) \ = \ v(c,a), \quad u(b) \ = \ v(c,b). \end{cases} \quad (4.27)$$

This problem clearly admits a unique viscosity solution, which must coincide with $v(c,\cdot)$ in $[a,b]$ by item 1. On the other hand, by since $\sigma^2(\cdot) > 0$, (4.27) is a uniformly elliptic problem, so it admits a solution of class $C^0([a,b];\mathbb{R}) \cap C^2((a,b);\mathbb{R})$, which is also a viscosity solution, so coincides with v. Hence, we deduce that $v(c,\cdot) \in C^2((\hat{d}_-(c), \hat{d}(c));\mathbb{R})$, and satisfies in a classical sense:

$$[\mathcal{L}v(c,\cdot)](d) - g(c,d) \quad = \quad 0, \qquad d \in (\hat{d}_-(c), \hat{d}_+(c)).$$

3. Notice that $\hat{V}(c,\cdot)$ is a particular solution to the ODE

$$[\mathcal{L}\phi(c,\cdot)](d) - g(c,d) \quad = \quad 0, \qquad d \in \mathcal{O}. \quad (4.28)$$

Therefore the general solution to (4.28) is in the form:

$$A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c,d), \qquad d \in S_c$$

for some real-valued constants $A(c)$, $B(c)$, which proves, together with item 2, the structure (4.15) of v in S_c.

The extension of (4.15) at $\hat{d}_-(c)$ and at $\hat{d}(c)$, when they do not coincide with $d_{\min}, d_{\max}$, respectively, can be obtained by taking $a = \hat{d}_-(c)$ and $b = \hat{d}_+(c)$ in the argument above. □

Lemma 4.2. *We have*

$$\lim_{d \downarrow d_{\min}} (v(c,d) - \hat{V}(c,d)) \quad = \quad 0, \quad \forall c \in (\underline{c}_+, \underline{c}_-); \quad (4.29)$$

$$\lim_{d \uparrow d_{\max}} (v(c,d) - \hat{V}(c,d)) \quad = \quad 0, \quad \forall c \in (\bar{c}_+, \bar{c}_-). \quad (4.30)$$

[4]The proof works even if the function is just upper semicontinuous.

Proof. We prove (4.29), the proof of (4.30) is analogous.

Fix $c \in (\underline{c}_+, \underline{c}_-)$. In this case we have $\hat{d}_-(c) = d_{\min}$. Then, due to Lemma 4.1, we have that $v(c, \cdot) \in C^2((d_{min}, \hat{d}_+(c)); \mathbb{R})$, and that it satisfies in a classical sense

$$[\mathcal{L}v(c, \cdot)](d) - g(c, d) \;=\; 0, \qquad \forall d \in (d_{\min}, \hat{d}_+(c)). \tag{4.31}$$

Let $d_0 \in (d_{\min}, \hat{d}_+(c))$ be fixed and take a generic $d \in (d_{\min}, d_0)$. Consider the stopping time

$$\tau_d \;=\; \inf\{t \geq 0 \mid D_t^d \geq d_0\}.$$

Since $d_{\min}$ is not-entrance for the diffusion D, we have (see e.g. [23, Ch. 20]):

$$\tau_d \nearrow \infty \quad \text{when} \quad d \downarrow d_{\min}. \tag{4.32}$$

Given a sequence $(d_n) \subset (d_{\min}, d)$ such that $d_n \downarrow d_{\min}$ consider the stopping times

$$\tau_d^n \;=\; \inf\{t \geq 0 \mid D_t^d \leq d_n\}.$$

Since $d_{\min}$ is inaccessible for the diffusion D, we have

$$\tau_d^n \nearrow \infty \quad \text{when} \quad n \to \infty. \tag{4.33}$$

By (4.31) and definition of τ_d, we apply Itô's formula to $v(c, D_t^d)$ in the interval $[0, \tau_d \wedge \tau_d^n \wedge n)$,

$$v(c,d) \;=\; \int_0^{\tau_d \wedge \tau_d^n \wedge n} e^{-\rho t} g(c, D_t^d)\mathrm{d}t + \int_0^{\tau_d \wedge \tau_d^n \wedge n} e^{-\rho t} v_d(c, D_t^d)\mathrm{d}W_t + e^{-\rho\tau_d} v(c, D^d_{\tau_d \wedge \tau_d^n \wedge n}).$$

By taking the expectation (noting that the expectation of the stochastic integral vanishes by our localization and that $v \geq 0$), we get

$$v(c,d) \;\geq\; \mathbb{E}\Big[\int_0^{\tau_d \wedge \tau_d^n \wedge n} e^{-\rho t} g(c, D_t^d)\mathrm{d}t\Big].$$

By taking the limit for $n \to \infty$ (note that $g \geq 0$, so we can use monotone convergence) and using (4.33), we get

$$v(c,d) \;\geq\; \mathbb{E}\Big[\int_0^{\tau_d} e^{-\rho t} g(c, D_t^d)\mathrm{d}t\Big].$$

Subtracting $\hat{V}(c,d)$ in both sides of the inequality above, we get

$$v(c,d) - \hat{V}(c,d) \;\geq\; \mathbb{E}\Big[\int_{\tau_d}^{\infty} e^{-\rho t} g(c, D_t^d)\mathrm{d}t\Big]$$

Taking the liminf for $d \downarrow d_{\min}$, and using (4.32), we obtain

$$\liminf_{d \downarrow d_{\min}} \, (v(c,d) - \hat{V}(c,d)) \;\geq\; 0,$$

and so the required limiting result, since we always have $v \leq \hat{V}$ (see (3.4)). $\square$

4.2 Structure of the value function

We can now provide the complete structure of the value function. Let us define

$$\mathcal{O}_+ \;:=\; \{d \in \mathcal{O} \mid \hat{c}_+(d) > -\infty\}, \qquad \mathcal{O}_- \;:=\; \{d \in \mathcal{O} \mid \hat{c}_-(d) < \infty\}.$$

Note that $\mathcal{O}_\pm$ are connected due to monotonicity of $\hat{c}_\pm$.

Theorem 4.1. *(Structure and properties of the value function)*
There exist functions

$$A, B \in \; C^1((\underline{c}_+, \bar{c}_-); \mathbb{R}), \qquad z_\pm : \mathcal{O}_\pm \to \mathbb{R},$$

(with A, B eventually extendable to C^1 functions up to $\underline{c}_+, \bar{c}_-$, respectively, when there exists $d \in \mathcal{O}$ such that $\hat{c}_+(d) = \underline{c}_+$, or when there exists $d \in \mathcal{O}$ such that $\hat{c}_-(d) = \bar{c}_-$), such that

$$v(c,d) \;=\; \begin{cases} A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c,d), & \text{on } \bar{\mathcal{C}}, \\ z_+(d) - q_0^+ c, & \text{on } \mathcal{A}^+, \\ z_-(d) + q_0^- c, & \text{on } \mathcal{A}^-. \end{cases} \tag{4.34}$$

Moreover:

(i) $A(c) \;=\; 0$ for every $c \in [\bar{c}_+, \bar{c}_-)$, and $B(c) \;=\; 0$ for every $c \in (\underline{c}_+, \underline{c}_-]$ (note that these intervals may be empty).

(ii) $z_\pm$ can be written in terms of the values of v at $\partial\mathcal{C}$ and of $\hat{c}_\pm$ as

$$z_+(d) \;=\; v(\hat{c}_+(d), d) + q_0^+ \hat{c}_+(d), \qquad d \in \mathcal{O}_+, \tag{4.35}$$
$$z_-(d) \;=\; v(\hat{c}_-(d), d) - q_0^- \hat{c}_-(d), \qquad d \in \mathcal{O}_-. \tag{4.36}$$

Proof. *Structure of v in $\bar{\mathcal{C}}$.* By Lemma 4.1(3), we already know that there exist functions $A, B : (\underline{c}_+, \overline{c}_-) \to \mathbb{R}$ such that we have

$$v(c,d) \;=\; A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c,d), \qquad (c,d) \in \mathcal{C}. \tag{4.37}$$

Let $c_0 \in (\underline{c}_+, \overline{c}_-)$. Since $\mathcal{C}$ is open, from the representation (4.10) we see that we can find $d, d_0 \in \mathcal{O}$ such that $(c, d_0), (c, d) \in S_c$ for every $c \in (c_0 - \varepsilon, c_0 + \varepsilon)$, for some $\varepsilon > 0$. Writing (4.37) at $(c,d), (c,d_0) \in \mathcal{C}$, and taking into account that $\psi(d)\varphi(d_0) - \varphi(d)\psi(d_0) \neq 0$ for all $d \neq d_0$ (this is due to strict monotonicity of φ, ψ), we can retrieve A, B in the interval $(c_0 - \varepsilon, c_0 + \varepsilon)$ as

$$A(c) \;=\; \frac{(v(c,d) - \hat{V}(c,d))\varphi(d_0) - (v(c,d_0) - \hat{V}(c,d_0))\varphi(d)}{\psi(d)\varphi(d_0) - \varphi(d)\psi(d_0)}, \tag{4.38}$$

$$B(c) \;=\; \frac{(v(c,d_0) - \hat{V}(c,d_0))\psi(d) - (v(c,d) - \hat{V}(c,d))\psi(d)}{\psi(d)\varphi(d_0) - \varphi(d)\psi(d_0)}. \tag{4.39}$$

Hence, since $v(\cdot, d)$ and $\hat{V}(\cdot, d)$ are of class C^1 for any fixed $d \in \mathcal{O}$, we get, by arbitrariness of c_0, that $A, B \in C^1((\underline{c}_+, \bar{c}_-); \mathbb{R})$.

Now assume that there exists $d \in \mathcal{O}$ such that $\hat{c}_+(d) = \underline{c}_+$. Then, since the function $\hat{c}_+$ is nondecreasing and right-continuous, there exists an interval $(a,b) \subset \mathcal{O}$ such that $\hat{c}_+(d) = \underline{c}_+$ in

(a, b). Take $d_0, d \in (a, b)$. Then, for every $c > \underline{c}_+$, it is $(c, d_0), (c, d) \in \mathcal{C}$. We can then write the relation (4.38) for every $c > \underline{c}_+$ and pass it to the limit for $c \downarrow \underline{c}_+$. In such a way we see that A can be extended to C^1 function up to $\underline{c}_+$. The same argument holds true for the other case involving B and $\bar{c}_-$.

Let us now check that (4.37) also holds at the points of the boundary $\partial\mathcal{C}$. Let $(c, d) \in \partial^+\mathcal{C}$. In this case, one of the following case must hold:

(a) $d = \hat{d}_+(c) \in \mathcal{O}$,

(b) $c = \hat{c}_+(d)$ and $\{(c, d) \mid c \in (\hat{c}_+(d), \hat{c}(d) + \varepsilon)\} \subset \mathcal{C}$ for some $\varepsilon > 0$,

(c) $d = \hat{d}_+(c')$ for $c' \in (c, c + \varepsilon)$ for some $\varepsilon > 0$.

In the case (a) the form (4.37) holds by Lemma 4.1 (3). In the case (b) the structure (4.37) holds by continuity of A, B and of v with respect to c, and by the already proved structure in $\mathcal{C}$. In the case (c) the structure (4.37) holds by case (a) and by continuity of A, B and of v with respect to c.

The same argument holds for points belonging to the boundary $\partial^-\mathcal{C}$, so we conclude that

$$v(c, d) \quad = \quad A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c, d), \qquad \text{in } \bar{\mathcal{C}}. \tag{4.40}$$

Structure of v in $\mathcal{A}^\pm$. This follows directly from the definition (4.1) of $\mathcal{A}^\pm$.

Let us now prove the remaining properties.

(i) Let $c \in (\bar{c}_+, \bar{c}_-)$. We can use (4.40) and write

$$\lim_{d \uparrow d_{\max}} v(c, d) \quad = \quad \lim_{d \uparrow d_{\max}} (A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c, d)).$$

By taking into account Lemma 4.2 and (3.8), we see that it must be $A(c) = 0$. In a similar way one proves that $B(c) = 0$ for every $c \in (\underline{c}_+, \underline{c}_-)$. Then $A(\bar{c}_+) = 0$ and $B(\underline{c}_-) = 0$ follow by continuity.

(ii) It follows using (4.34) and by evaluating v at the points $(\hat{c}_\pm(d), d) \in \bar{\mathcal{C}}$. □

4.3 Optimal control

In the following we suppress, for simplicity of notation, the superscript d in D^d. Moreover, the superscript k in the notation C_t^k below will not denote the initial datum, but a running natural index.

Let $(c, d) \in \mathcal{S}$. Let us define, with the convention $\inf \emptyset = \infty$, the random times

$$\tau_0^+ \;\; := \;\; \inf\,\{t \geq 0 \mid c < \hat{c}_+(D_t)\}, \qquad \tau_0^- \;\; := \;\; \inf\,\{t \geq 0 \mid c > \hat{c}_-(D_t)\}, \qquad \tau_0 \;\; := \;\; \tau_0^+ \wedge \tau_0^-.$$

Due to (4.5), we have $\{\tau_0^+ = \tau_0^-\} = \{\tau_0 = \infty\}$. Define also

$$\Omega_\infty \;\; := \;\; \{\tau_0 = \infty\}, \qquad \Omega_+ \;\; := \;\; \{\tau_0^+ < \tau_0^-\}, \qquad \Omega_- \;\; := \;\; \{\tau_0^+ > \tau_0^-\}.$$

Define

$$C_t^0 \quad = \quad c, \qquad t \geq 0,$$

and define recursively the following processes and stopping times:

- For all $k \geq 0$,

$$\overline{D}_t^k \ := \ \max_{s\in[\tau_{k-1},t]} D_s, \qquad \underline{D}_t^k \ := \ \min_{s\in[\tau_{k-1},t]} D_s, \qquad t \geq \tau_{k-1},$$

- If $k \geq 1$ is odd,

$$C_t^k \ := \ \begin{cases} c, & \text{on } \Omega_\infty, \\ c + \hat{c}_+(\overline{D}_t^k), & \text{on } \Omega_+, \\ c + \hat{c}_-(\underline{D}_t^k), & \text{on } \Omega_-, \end{cases} \qquad t \ \geq \ \tau_{k-1},$$

$$\tau_k \ := \ \begin{cases} \infty, & \text{on } \Omega_\infty, \\ \inf\ \{t \geq \tau_{k-1} \mid C_t^{*,k} > \hat{c}_-(D_t)\}, & \text{on } \Omega_+, \\ \inf\ \{t \geq \tau_{k-1} \mid C_t^{*,k} < \hat{c}_+(D_t)\}, & \text{on } \Omega_-. \end{cases}$$

- If $k \geq 2$ is even

$$C_t^k \ := \ \begin{cases} c, & \text{on } \Omega_\infty, \\ c + \hat{c}_+(\overline{D}_t^k), & \text{on } \Omega_-, \\ c + \hat{c}_-(\underline{D}_t^k), & \text{on } \Omega_+, \end{cases} \qquad t \ \geq \ \tau_{k-1},$$

$$\tau_k \ := \ \begin{cases} \infty, & \text{on } \Omega_\infty, \\ \inf\ \{t \geq \tau_{k-1} \mid C_t^{*,k} > \hat{c}_-(D_t)\}, & \text{on } \Omega_-, \\ \inf\ \{t \geq \tau_{k-1} \mid C_t^{*,k} < \hat{c}_+(D_t)\}, & \text{on } \Omega_+. \end{cases}$$

Since $\mathcal{A}^\pm$ are closed and $\sigma^2 > 0$, we have, if k is odd

$$\begin{aligned} \inf\{t \geq \tau_k \mid (C_t^{*,k}, D_t) \in \overset{\circ}{\mathcal{A}}{}^+\} \ &= \ \inf\{t \geq \tau_k \mid C_t^{*,k} < \hat{c}_+(D_t)\}, \ \text{ a.e. in } \Omega_+, \\ \inf\{t \geq \tau_k \mid (C_t^{*,k}, D_t) \in \overset{\circ}{\mathcal{A}}{}^-\} \ &= \ \inf\{t \geq \tau_k \mid C_t^{*,k} > \hat{c}_-(D_t)\}, \ \text{ a.e. in } \Omega_-, \end{aligned}$$

and similar representations if k is even. Hence, since $\mathbb{F}$ satisfies the usual conditions, so hitting times of open sets are stopping times, we see that the sequence (τ_k) is a sequence of stopping times. Setting $\tau_{-1} := 0$, define the process

$$C_t^* \ := \ \sum_{k=0}^{\infty} C_t^k \, \mathbf{1}_{[\tau_{k-1},\tau_k)}(t), \qquad t \geq 0. \tag{4.41}$$

Since $\tau_k \to \infty$ almost surely, the process C^* is well defined for every $t \geq 0$. Moreover it is clearly right-continuous and adapted. By construction

$$(C_t^*, D_t) \ \in \ \bar{\mathcal{C}}, \qquad \forall t \geq 0. \tag{4.42}$$

Define the control

$$I_t^* \ := \ C_t^* - c. \tag{4.43}$$

The control process I^* does the minimum effort to keep the couple (C_t^*, D_t) inside $\bar{\mathcal{C}}$. More precisely, at time $t \geq 0$:

- if $(C^*_{t-}, D_t) \in \mathcal{C}$, no action is taken ($\mathrm{d}I^* = 0$);
- if $(C^*_{t-}, D_t) \in \partial\mathcal{C}$ (e.g., assume $(C^*_{t-}, D_t) \in \partial^+\mathcal{C}$; simmetrically one can argue in the case $(C^*_{t-}, D_t) \in \partial^-\mathcal{C}$), then two cases have to be distinguished:
  - if $C^*_{t-} = \hat{c}_+(D_t)$ (which occurs in particular if $\hat{c}$ is continuous at D_t), then I^* acts in order to reflect (C^*_t, D_t) at the boundary $\partial\mathcal{C}^+$ along the positive c-direction. Note that no action is taken if $\hat{c}_+$ is constant in a right-neighborhood of D_t.
  - if $\hat{c}_+$ is discontinuous at D_t and $C^*_{t-} < \hat{c}_+(D_t)$, then the process C^* has a positive jump $\Delta C^*_t = \Delta I^{*,+}_t = \hat{c}_+(D_t) - C^*_{t-}$.

Regarding the last possibility, letting $\mathcal{N}^\pm$ be the (at most countable) sets of discontinuity points of $\hat{c}_\pm$, respectively, due to the continuity of trajectories of D, we see that the process $I^* = I^{*,+} - I^{*,-}$ can jump

(a.1) either at time 0 when $c < \hat{c}_+(d)$ or when $c > \hat{c}_-(d)$, and in this case we have, respectively, $\Delta I^*_0 = \Delta I^{*,+}_0 = \hat{c}_+(d) - c$ or $\Delta I^*_0 = -\Delta I^{*,-}_0 = \hat{c}_-(d) - c$;

(a.2) when $D_t \in \mathcal{N}^+$ and $C^*_{t-} < \hat{c}_+(D_t)$, and in this case $\Delta I^*_t = \Delta I^{*,+}_t = \hat{c}_+(D_t) - C^*_{t-}$.

(a.3) when $D_t \in \mathcal{N}^-$ and $C^*_{t-} > \hat{c}_-(D_t)$, and in this case $\Delta I^*_t = -\Delta I^{*,-}_t = C^*_{t-} - \hat{c}_-(D_t)$.

Lemma 4.3. *The processes C^*, I^* satisfy*

$$\int_0^\infty e^{-\rho t} \mathbf{1}_{\{(C^*_t, D_t) \in \mathcal{C}\}} \, \mathrm{d}I^{*,\pm}_t \;=\; 0. \tag{4.44}$$

Proof. Fix $\omega \in \Omega$ and suppose that $(C^*_t(\omega), D^d_t(\omega)) \in \mathcal{C}$. Then, by definition of the τ_k's and since $\mathcal{C}$ is open, we must have $t \in (\tau_{k-1}(\omega), \tau_k(\omega))$ for some $k \geq 0$, and

$$C^*_t(\omega) \;\in\; \big(\hat{c}_+(D_t(\omega)),\ \hat{c}_-(D_t(\omega))\big). \tag{4.45}$$

By definition of C^*, τ_{k-1}, τ_k, we see that $C.^*(\omega)$ is constant in some suitable neighborhood $(t - \varepsilon(\omega), t + \varepsilon(\omega))$ of t, hence also $I.^*(\omega)$ is constant therein. Thus, we have proved (4.44). □

The second main result provides the existence and an explicit description of the optimal state process (and a description of the optimal investment in terms of the optimal state).

Theorem 4.2. *(Optimal control) Let $(c,d) \in \mathcal{S}$. The process C^* constructed before in* (4.41) *is an optimal state process for the value function at (c,d), with corresponding optimal control $I^* = (I^{*,+}, I^{*,-})$ defined by* (4.43).

Proof. Let us show that

$$v(c,d) \;\geq\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(g(C^*_t, D_t) + q_0^+ \mathrm{d}I^{*,+}_t - q_0^- \mathrm{d}I^{*,-}_t\Big)\Big]. \tag{4.46}$$

Let (K_n) be an increasing sequence of compact subsets of $\mathcal{S}$ such that $\cup_{n\in\mathbb{N}} K_n = \mathcal{S}$. Consider the (bounded) stopping time $\tau_n = \inf\{t \geq 0 \mid C^*_t \wedge D_t \notin K_n\} \wedge n$, and notice that $\tau_n \nearrow \infty$ a.s. when n goes to infinity. From (4.40) and since $\hat{V} \in C^{1,2}(\mathcal{S};\mathbb{R})$, we see that $v \in C^{1,2}(\bar{\mathcal{C}};\mathbb{R})$. Thus, by (4.42), we may apply Itô's formula (see Proposition A.4) to $e^{-\rho t} v(C^*_t, D^d_t)$ between

0 and τ_n, take expectation, and obtain (after observing that the stochastic integral over the interval $[0, \tau_n \wedge T)$ vanishes in expectation due to our localization):

$$\begin{aligned} v(c,d) \;&=\; \mathbb{E}\Big[e^{-\rho\tau_n} v(C^*_{\tau_n\wedge T}, D_{\tau_n\wedge T})\Big] \;+\; \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}[\mathcal{L}v(C^*_t,\cdot)](D_t)\mathrm{d}t\Big] \\ &\quad - \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t} v_c(C^*_t, D_t)\mathrm{d}I^*_t\Big] \\ &\quad - \mathbb{E}\Big[\sum_{0\le t\le\tau_n} e^{-\rho t}(v(C^*_t, D_t) - v(C^*_{t-}, D_t) - v_c(C^*_t, D_t)\Delta C^*_t)\Big], \end{aligned} \tag{4.47}$$

Now observe that $[\mathcal{L}v(c',\cdot)](d') = g(c', d')$ for (c', d') in $\mathcal{C}$ but also in $\bar{\mathcal{C}}$ by continuity of g and since $v \in C^{1,2}(\bar{\mathcal{C}};\mathbb{R})$. This implies

$$\mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}[\mathcal{L}v(C^*_t,\cdot)](D^d_t)\mathrm{d}t\Big] \;=\; \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}\big(g(C^*_t, D^d_t)\mathrm{d}t\Big]. \tag{4.48}$$

Now, notice that $\mathrm{d}I^{*,+} = 0$ if $(C^*_t, D^d_t) \in \mathcal{A}^-$ and $\mathrm{d}I^{*,-} = 0$ if $(C^*_t, D^d_t) \in \mathcal{A}^+$. Then taking into account (4.44) and the fact that $v_c = -q^+_0$ in $\mathcal{A}^+$ and $v_c = q^-_0$ in $\mathcal{A}^-$, we have

$$-\,\mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t} v_c(C^*_t, D^d_t)\mathrm{d}I^*_t\Big] \;=\; \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}(q^+_0\mathrm{d}I^{*,+}_t + q^-_0\mathrm{d}I^{*,-}_t)\Big]. \tag{4.49}$$

Moreover, considering the three possibilities of jump (a.1)–(a.3) described above for I^*, we have

$$v(C^*_t, D^d_t) - v(C^*_{t-}, D^d_t) - v_c(C^*_t, D^d_t)\Delta C^*_t \;=\; 0, \quad \forall t \ge 0. \tag{4.50}$$

Therefore by nonnegativity of v and (4.47)--(4.50), we have

$$v(c,d) \;\ge\; \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}\big(g(C^*_t, D^d_t)\mathrm{d}t + q^+_0\mathrm{d}I^{*,+}_t + q^-_0\mathrm{d}I^{*,-}_t\big)\Big].$$

Letting $n \to \infty$, from monotone convergence we get the inequality (4.46). Since the opposite inequality always holds by definition of v, this proves the equality, i.e. that I^* is an optimal control. □

The picture below represents a possible shape of the solution. The state space region $\mathcal{S}$ is the half-plane on the right of the vertical dotted line. When the system lies in the continuation region $\mathcal{C}$, it moves along the horizontal lines and no action is taken. Whenever the system touches the boundary $\partial\mathcal{C}$, the optimal control (acting along the vertical lines as indicated by the arrows in the picture) consists in doing the minimal effort to keep the system in $\bar{\mathcal{C}}$. We notice that, if the boundary $\hat{c}_+$ or the boundary $\hat{c}_-$ is constant somewhere, no action is taken if the system reaches this part of boundary, and the system lies on this part of the boundary for a certain time until it meets a strictly increasing part of this boundary.

Remark 4.2. From the solution found, it turns out that when $\underline{c}_- \ge 0$, starting from $c \ge 0$ the optimal state process verifies $C^* \ge 0$. This means that the solution is, henceforth, also thee solution of the problem with state constraint $C \ge 0$.

Corollary 4.1. *1. If $\lim_{c\downarrow-\infty} g_c(c,d) = -\infty$, then $\hat{c}_+ > -\infty$ in $(d, d_{\max})$.*

2. If $\lim_{c\uparrow\infty} g_c(c,d) = \infty$*, then* $\hat{c}_- < \infty$ *in* $(d_{\min}, d)$.

Proof. We prove item 1, then item 2 can be proved symmetrically.

Let $d \in \mathcal{O}$ be such that $\lim_{c\downarrow-\infty} g_c(c,d) = -\infty$. Take $c_0 \in \mathbb{R}$ such that $g_c(c_0,d) \le 0$ and $\hat{c}_-(d) > c_0$. Since by Assumption 2.2 g_c is nondecreasing in c and nonincreasing in d, we have $g_c \le 0$ in $(-\infty, c_0] \times [d, d_{\max})$. Assume, by contradiction, that there exists $d_1 \in (d, d_{\max})$ such that $\hat{c}_+(d_1) = -\infty$. By monotonicity of $\hat{c}_+$ this implies that $\hat{c}_+ \equiv -\infty$ in $(d_{\min}, d_1]$. Now, given any $c \le c_0$ and $d_0 \in (d, d_1)$, define the stopping times

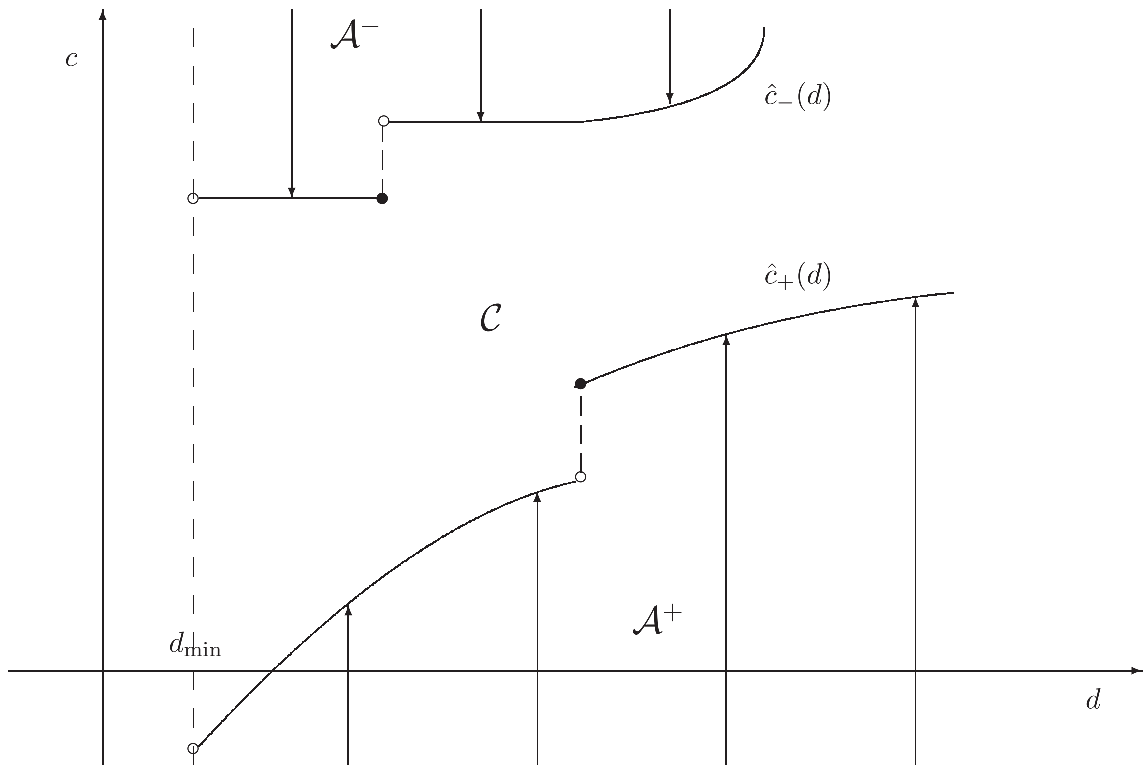
$$\sigma = \inf\{t \ge 0 \mid D_t^{d_0} \le d\}, \quad \tau = \inf\{t \ge 0 \mid D_t^{d_0} \ge d_1\}, \quad \tau^*(c) = \inf\{t \ge 0 \mid D_t^{d_0} \ge \hat{d}_+(c)\}.$$

Observe that $\tau \le \tau^*(c)$, for every $c \in \mathbb{R}$, since $\hat{d}_+(c)$ has to be larger than d_1, as $\hat{c}_+ \equiv -\infty$ in $(d_{\min}, d_1]$. Moreover, by Proposition 3.3 and Theorem 4.2, $\tau^*(c)$ is the optimal stopping time of P2 for the Dynkin game defined in Subsection 3.2. Hence, we must have, taking also into account that $g_c(c,\cdot)$ is nonincreasing, that $g_c \le 0$ in $(-\infty, c_0] \times [d, d_{\max})$, and that $\tau \le \tau^*(c)$,

$$\begin{aligned} v_c(c,d) &\le J(c,d;\sigma,\tau^*(c)) \\ &= \mathbb{E}\left[\int_0^{\tau^*(c)\wedge\sigma} e^{-\rho t} g_c(c, D_t^{d_0})\mathrm{d}t + q_0^- e^{-\rho\sigma}\mathbf{1}_{\{\sigma<\tau^*(c)\}} - q_0^+ e^{-\rho\tau^*(c)}\mathbf{1}_{\{\tau^*(c)<\sigma\}}\right] \\ &\le \mathbb{E}\left[\int_0^{\tau\wedge\sigma} e^{-\rho t} g_c(c, D_t^{d_0})\mathrm{d}t + q_0^-\right] \\ &\le \mathbb{E}\left[\int_0^{\tau\wedge\sigma} e^{-\rho t} g_c(c, d)\mathrm{d}t + q_0^-\right] \\ &= \frac{g_c(c,d)}{\rho}\,\mathbb{E}[1 - e^{-\rho(\tau\wedge\sigma)}] + q_0^-. \end{aligned}$$

Note that σ and τ are independent of c, and that $\tau\wedge\sigma > 0$. So, letting $c \to -\infty$ in the inequality above we get $\lim_{c\to-\infty} v_c(c,d) = -\infty$, which contradicts Proposition 3.4 (3). □

Remark 4.3. We notice that items 1 and 2 of Corollary 4.1 above hold, respectively, when $q_0^+ < \infty$ and $q_0^- < \infty$, which is an assumption we are doing throughout the paper. However, also referring to Remark 2.2 (2), we point out that in the case one consider, e.g., $q_0^- = \infty$ (irreversible investment), one has immediately $\hat{c}_- \equiv \infty$, so Corollary 4.1 does not hold anymore.



5 Quadratic cost: smooth fit and boundaries' characterization

Theorem 4.1 and the continuity of v_c in $\mathcal{S}$ yield some optimality conditions. Indeed, we should have

$$\begin{cases} A'(c)\psi(d) + B(c)\varphi(d) + \hat{V}_c(c,d) = -q_0^+, & \forall\,(c,d) \in \partial\mathcal{C}^+, \\ \\ A'(c)\psi(d) + B(c)\varphi(d) + \hat{V}_c(c,d) = q_0^-, & \forall\,(c,d) \in \partial\mathcal{C}^-. \end{cases} \tag{5.1}$$

It is clear that one cannot expect that the conditions above provide a way either to find the value function or the optimal boundaries $\partial^{\pm}\mathcal{C}$ (e.g., in terms of the functions $\hat{c}_{\pm}$), as, read at $(\hat{c}_{\pm}(d), d)$, they would relate four unknown functions $A, B, \hat{c}_{\pm}$ by two equations. Other optimality conditions are needed and should be derived from some other suitable smoothness property of the value function at the optimal boundaries $\partial^{\pm}\mathcal{C}$. To this end, we notice by Theorem 4.1 that

$$\frac{\partial}{\partial d} v_c(c,d) = 0 \;\text{ in } \mathcal{A}^{\pm}. \tag{5.2}$$

Therefore, a requirement of a smooth fit condition of the second order mixed derivative of v at the optimal boundaries would imply

$$\lim_{(c,d)\to(c_0,d_0)} v_{cd}(c,d) = 0, \quad \forall\,(c_0,d_0) \in \partial^{\pm}\mathcal{C}. \tag{5.3}$$

This is what we are going to prove in the next subsection under further assumptions on g.

5.1 The smooth fit-principle

The purpose of the present subsection is indeed to prove (5.3). However, we need to further specify our assumptions, restricting to the quadratic cost case:

$$g(c,d) \;=\; \frac{1}{2}(c^2 - 2\beta_0(d)c + \alpha_0(d)), \tag{5.4}$$

where α_0, β_0 are continuous functions. From now on, we assume that g has the structure (5.4) and we do not repeat this assumption in the statements of the results. We assume that the functions α_0, β_0 are continuous and that β_0 is nondecreasing, so that Assumption 2.2 holds true, and we denote

$$\alpha(d) \;:=\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\alpha_0(D_t^d)\mathrm{d}t\Big], \qquad \beta(d) \;:=\; \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\beta_0(D_t^d)\mathrm{d}t\Big], \tag{5.5}$$

noting that $\alpha, \beta \in C^2(\mathcal{O};\mathbb{R})$ as the diffusion D is nondegenerate. The function $\hat{V}$ is written in this case as:

$$\hat{V}(c,d) \;=\; \frac{1}{2}\Big(\frac{1}{\rho}c^2 - 2\beta(d)c + \alpha(d)\Big). \tag{5.6}$$

Given a function $\varphi \in C(\mathbb{R};\mathbb{R})$, let us denote

$$[\Delta^2\varphi](x;\varepsilon) \;:=\; \frac{1}{\varepsilon^2}[\varphi(x+\varepsilon) + \varphi(x-\varepsilon) - 2\varphi(x)], \quad x \in \mathbb{R},\ \varepsilon > 0.$$

The following Lemma, which relies on assumption (5.4), enables us to obtain further regularity of the value function with respect to c (Corollary 5.1), which is crucial to prove then (5.3).

Lemma 5.1. *We have for every $(c,d) \in \mathcal{S},\ \varepsilon > 0$,*

$$0 \;\le\; [\Delta^2 v(\cdot,d)](c;\varepsilon) \;\le\; \frac{1}{\rho}.$$

Proof. The estimate from below is a straightforward consequence of the convexity of v with respect to c. Let us prove the estimate from above. Let $(c,d) \in \mathcal{S}$, $\varepsilon > 0$, and $I \in \mathcal{I}$. By using the fact that $g_{cc} \equiv 1$ under (5.4), we have

$$\begin{aligned} & \frac{1}{\varepsilon^2}\left[G(c+\varepsilon,d;I) + G(c-\varepsilon,d;I) - 2G(c,d;I)\right] \\ =\; & \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big[\frac{1}{\varepsilon^2}\big(g(C_t^{c+\varepsilon,I},D_t^d) + g(C_t^{c-\varepsilon,I},D_t^d) - 2g(C_t^{c,I},D_t^d)\big)\Big)\mathrm{d}t\Big] \;=\; \frac{1}{\rho}. \end{aligned} \tag{5.7}$$

Since

$$v(c+\varepsilon,d) + v(c-\varepsilon,d) - 2G(c,d;I) \;\le\; G(c+\varepsilon,d;I) + G(c-\varepsilon,d;I) - 2G(c,d;I),$$

we get from (5.7):

$$\frac{1}{\varepsilon^2}\left[v(c+\varepsilon,d) + v(c-\varepsilon,d) - 2G(c,d;I)\right] \;\le\; \frac{1}{\rho}, \qquad \forall I \in \mathcal{I}.$$

Taking the supremum over $I \in \mathcal{I}$, this proves the required upper-estimate. □

Lemma 5.1 implies that $v_c(\cdot,d)$ is Lipschitz continuous for each $d \in \mathcal{O}$. Together with (4.38)-(4.39) and (5.6), we immediately get the following regularity result.

Corollary 5.1. *The derivative functions $A', B' : (\underline{c}_+, \bar{c}_-) \to \mathbb{R}$, where A, B are the functions defined in Theorem 4.1, are locally Lipschitz. In other terms $A, B \in W^{2,\infty}_{loc}((\underline{c}_+, \bar{c}_-); \mathbb{R})$.*

(This property holds eventually up to $\underline{c}_+, \bar{c}_-$, when A, B can be extended, respectively, to C^1 functions up to $\underline{c}_+, \bar{c}_-$, according to the conditions of Theorem 4.1 which allow these extensions.)

We are now able to prove the second order smooth-fit result on the value function.

Proposition 5.1. *The relation* (5.3) *hold true.*

Proof. Since $v_{cd} = 0$ in $\mathcal{A}^{\pm}$, the claim is equivalent to prove that

$$\lim_{\substack{(c,d)\to(c_0,d_0)\\ (c,d)\in\mathcal{C}}} v_{cd}(c,d) \;=\; 0, \qquad \forall\,(c_0,d_0)\in\partial^{\pm}\mathcal{C}. \tag{5.8}$$

We shall prove (5.8) for the lower boundary $\partial^+\mathcal{C}$; the claim concerning the upper boundary $\partial^-\mathcal{C}$ can be proved in the same way. Letting $(c_0, d_0) \in \partial^+\mathcal{C}$ we distinguish three cases.

1. Suppose that $c_0 = \hat{c}_+(d) > \underline{c}_+$. Let us consider the function on $\mathcal{D} := (\underline{c}_+, \bar{c}_-) \times \mathcal{O}$

$$\bar{v}(c,d) \;:=\; A(c)\psi(d) + B(c)\varphi(d) + \hat{V}(c,d), \qquad (c,d)\in\mathcal{D}.$$

By Theorem 4.1 and (5.6), we have that $\bar{v} \in C^{1,2}(\mathcal{D};\mathbb{R})$, and that $\bar{v}_{cd}$ exists and is continuous in $\mathcal{D}$. Since $\bar{v} = v$ in $\bar{\mathcal{C}} \cap \mathcal{D}$, by monotonicity of $v_c(c,\cdot)$, we have

$$\bar{v}_{cd} \;\le\; 0 \quad \text{in } \mathcal{C}. \tag{5.9}$$

Clearly (5.8) is equivalent to

$$\lim_{\substack{(c,d)\to(c_0,d_0)\\ (c,d)\in\mathcal{C}}} \bar{v}_{cd}(c,d) \;=\; 0, \qquad \forall\,(c_0,d_0)\in\partial^{+}\mathcal{C}. \tag{5.10}$$

By continuity of $\bar{v}_{cd}$, the limit above exists and coincides with $\bar{v}_{cd}(c_0, d_0)$. Taking into account (5.9), suppose by contradiction that

$$\bar{v}_{cd}(c_0,d_0) \;<\; 0. \tag{5.11}$$

Then, by continuity of $\bar{v}_{cd}$, we may find $\varepsilon > 0$, $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\bar{v}_{cd}(c,d) \;\le\; -\varepsilon, \quad \forall (c,d)\in(c_0-\delta_1, c_0-\delta_1)\times(d_0-\delta_2, d_0+\delta_2)\subset\mathcal{D}. \tag{5.12}$$

Since $\bar{v}_c(c_0, d_0) = -q_0^+$, due to (5.11) and to Corollary 5.1, we can apply Implicit Function Theorem in a generalized form, stating that there exists $\hat{d}'_+$ in Sobolev sense in the interval $(c_0 - \delta_1, c_0 + \delta_1)$, and, assuming without loss of generality that

$$d_0-\delta_2 \;=\; \hat{d}_+(c_0-\delta_1), \qquad d_0+\delta_2 \;=\; \hat{d}_+(c_0+\delta_1), \tag{5.13}$$

that it holds, by Corollary 5.1 and (5.12)

$$\hat{d}'_+(\cdot) \;=\; -\frac{\bar{v}_{cc}(\cdot,\hat{d}_+(\cdot))}{\bar{v}_{cd}(\cdot,\hat{d}_+(\cdot))} \;\le\; M_\varepsilon \;<\; \infty, \quad \text{a.e. in } (c_0-\delta_1, c_0+\delta_1).$$

Let us now assume, without loss of generality (recall that $\hat{c}_+$ is right-continuous), that $\hat{c}_+$ is continuous on $[d_0, d_0 + \delta_2)$. Then, combining with (5.13)-(5.14), we see that $\hat{c}_+$ is strictly increasing on $[d_0, d_0 + \delta_2)$, there exists the inverse $\hat{c}_+^{-1}$ on $[c_0, c_0 + \delta_1)$, it coincides with $\hat{d}_+$, and $\hat{d}_+$ is continuous and strictly increasing as well on $[c_0, c_0 + \delta_1)$. It follows that

$$\hat{d}'_+ \;\; > \;\; 0, \quad \text{a.e. in} \;\; [c_0, c_0 + \delta_1). \tag{5.14}$$

Let $\mathcal{Y}$ be the set of differentiability points of $\hat{d}_+$ in $[d_0, d_0 + \delta)$ where $0 < \hat{d}'_+ < M_\varepsilon$. Then, taking into account (5.14)-(5.14), we see that $\mathcal{Y}$ has full measure in $[c_0, c_0 + \delta_1)$. Consequently $\hat{d}(\mathcal{Y})$ is dense in $[d_0, d_0 + \delta_2)$, $\hat{c}'_+$ exists in $\hat{d}_+(\mathcal{Y})$, and

$$\hat{c}'_+ \;\; \in \;\; [1/M_\varepsilon, \infty), \qquad \text{in} \;\; \hat{d}_+(\mathcal{Y}). \tag{5.15}$$

Let us now consider the function $d \in [d_0, d_0 + \delta_2) \mapsto v(c_0, d)$. Since $\hat{c}_+$ is nondecreasing in $[d_0, d_0 + \delta_2)$ (actually we have shown strictly increasing), the segment $\{(c_0, d) \mid d \in [d_0, d_0 + \delta_2)\}$ is contained in $\mathcal{A}^+$. Hence, Theorem 4.1 yields

$$v(c_0, d) \;\; = \;\; -q_0^+ c_0 + z_+(d), \qquad \forall d \in [d_0, d_0 + \delta). \tag{5.16}$$

Applying the chain rule at the points of $\hat{d}_+(\mathcal{Y})$ to

$$[d_0, d_0 + \delta) \;\to\; \mathbb{R}, \qquad d \;\mapsto\; z_+(d) \;=\; v(\hat{c}_+(d), d) + q_0^+ \hat{c}_+(d) \;=\; \bar{v}(\hat{c}_+(d), d) + q_0^+ \hat{c}_+(d),$$

we see that the function z_+ is differentiable at the points of $\hat{d}_+(\mathcal{Y})$ and

$$z'_+(d) \;\; = \;\; \bar{v}_c(\hat{c}_+(d), d)\hat{c}'_+(d) + \bar{v}_d(\hat{c}_+(d), d) + q_0^+ \hat{c}'_+(d), \quad \forall d \in \hat{d}_+(\mathcal{Y}).$$

By definition of $\hat{c}_+$, we have $\bar{v}_c(\hat{c}_+(d), d) = v_c(\hat{c}_+(d), d) = -q_0^+$ for every $d \in \mathcal{O}$, and so

$$z'_+(d) \;=\; v_d(\hat{c}_+(d), d), \quad \forall d \in \hat{d}_+(\mathcal{Y}).$$

Together with (5.16), this shows the existence of $v_d(c_0, d)$ for each $d \in \hat{d}_+(\mathcal{Y})$ and the equality

$$v_d(c_0, d) \;=\; z'_+(d) \;=\; \bar{v}_d(\hat{c}_+(d), d), \quad \forall d \in \mathcal{Y}. \tag{5.17}$$

On the other hand, by using again the chain rule, we can get from (5.17) the existence of $v_{dd}(c_0, d)$ for each $d \in \hat{d}_+(\mathcal{Y})$ and the equality

$$v_{dd}(c_0, d) \;=\; z''_+(d) \;=\; \bar{v}_{dd}(\hat{c}_+(d), d) + \bar{v}_{cd}(\hat{c}_+(d), d)\, \hat{c}'_+(d), \quad \forall d \in \hat{d}_+(\mathcal{Y}). \tag{5.18}$$

Therefore, from (5.12), (5.15), and (5.18), we get

$$v_{dd}(c_0, d) \;\; \leq \;\; \bar{v}_{dd}(\hat{c}_+(d), d) - \varepsilon/M_\varepsilon, \quad \forall d \in \hat{d}_+(\mathcal{Y}). \tag{5.19}$$

Now the viscosity subsolution property of v, and (5.16), (5.17), (5.19) yield

$$\begin{aligned} g(c_0, d) \;\; &\geq \;\; \rho v(c_0, d) - \mu(d) v_d(c_0, d) - \frac{1}{2}\sigma(d)^2 v_{dd}(c_0, d) \qquad (5.20) \\ &= \;\; \rho v(c_0, d) - \mu(d)\bar{v}_d(\hat{c}_+(d), d) - \frac{1}{2}\sigma(d)^2[\bar{v}_{dd}(\hat{c}_+(d), d) - \varepsilon/M_\varepsilon], \quad \forall d \in \hat{d}_+(\mathcal{Y}). \end{aligned}$$

Taking a sequence $(\alpha_n) \subset \hat{d}_+(\mathcal{Y})$ such that $\alpha_n \downarrow d_0$ (this can be done since $\hat{d}_+(\mathcal{Y})$ is dense in $[d_0, d_0+\delta_2))$ and passing to the limit in (5.20) evaluated at $d = \alpha_n$ we obtain by continuity of $\hat{c}_+$ in $[d_0, d_0+\delta_2)$, continuity of g in $\mathcal{S}$, and since $\bar{v} \in C^{1,2}(\mathcal{D}, \mathbb{R})$ anfd $\bar{v} = v$ in $\bar{\mathcal{C}}$,

$$\rho\bar{v}(c_0, d_0) - \mu(d_0)\bar{v}_d(c_0, d_0) - \frac{1}{2}\sigma(d_0)^2[\bar{v}_{dd}(c_0, d_0) - \varepsilon/M_\varepsilon] \quad \leq \quad g(c_0, d_0). \tag{5.21}$$

On the other hand, recall that $\mathcal{L}\bar{v} = \mathcal{L}v = g$ on $\mathcal{C}$. Therefore, since $v \in C^{1,2}(\mathcal{D}; \mathbb{R})$ and since $(c_0, d_0) \in \bar{\mathcal{C}}$, by continuity we must also have

$$\rho\bar{v}(c_0, d_0) - \mu(d_0)\bar{v}_d(c_0, d_0) - \frac{1}{2}\sigma(d_0)^2\bar{v}_{dd}(c_0, d_0) \quad = \quad g(c_0, d_0),$$

which is in contradiction with (5.21) as $\sigma^2(d_0) > 0$, and the claim is proved in this case.

2. Consider now the case $c_0 = \hat{c}_+(d_0) = \underline{c}_+$. In this case we can construct the function $\hat{v}$ in $\mathcal{D} := (\underline{c}_+ - \varepsilon, \bar{c}_-) \times \mathcal{O}$ for some $\varepsilon > 0$ by using the extension part of Corollary 5.1, and repeat the argument of the previous case.

3. Consider now the last possible case, i.e. $d_0 = \hat{d}_+(c_0)$ and $c_0 < \hat{c}_+(d_0)$, noting that $\hat{c}_+(d_0) < \infty$ (see Proposition 4.1 (1)). In this case the segment $K := \{(c, d_0) \mid c \in [c_0, \hat{c}_+(d_0)]\}$ is contained in $\partial^+\mathcal{C}$. Define the function $\bar{v}$ as in item 1. We then have $\bar{v}_c = v_c = -q_0^+$ in K. Hence

$$\begin{aligned} -q_0^+ - \bar{v}_c(c, d) &= \bar{v}_c(c, d_0) - \bar{v}_c(c, d) \\ &= \int_d^{d_0} \bar{v}_{cd}(c, \xi)\mathrm{d}\xi, \qquad \forall c \in [c_0, \hat{c}_+(d_0)],\ \forall d \leq d_0, \end{aligned} \tag{5.22}$$

Taking into account Corollary 5.1 and differentiating (5.22) with respect to c we get (the derivatives A'', B'' must be intended in Sobolev sense)

$$-\bar{v}_{cc}(c, d) \quad = \quad \int_d^{d_0} \bar{v}_{cdc}(c, \xi)\mathrm{d}\xi, \qquad \text{a.e. } (c, d) \in [c_0, \hat{c}_+(d_0)] \times (\hat{d}_-(c), d_0]. \tag{5.23}$$

Since $v_{cc} \geq 0$, hence $\bar{v}_{cc} \geq 0$ (in Sobolev sense), from (5.23) we get

$$0 \quad \geq \quad \int_d^{d_0} \bar{v}_{cdc}(c, \xi)\mathrm{d}\xi, \qquad \text{a.e. } (c, d) \in [c_0, \hat{c}_+(d_0)] \times (\hat{d}_-(c), d_0], \tag{5.24}$$

from which, taking into account (5.6), we deduce that actually

$$A''(c)\psi'(d) + B''(c)\varphi'(d) \quad \leq \quad 0, \qquad \text{a.e. in } [c_0, \hat{c}_+(d_0)] \times (\hat{d}_-(c), d_0],$$

Then, since ψ', φ' are continuous, we deduce that

$$A''(c)\psi'(d_0) + B''(c)\varphi'(d_0) \quad \leq \quad 0, \qquad \text{a.e. in } [c_0, \hat{c}_+(d_0)].$$

Hence, $\bar{v}_{cd}(\cdot, d_0)$ is nonincreasing with respect to c in $[c_0, \hat{c}_+(d_0)]$. Then, assuming now, as in item 1, by contradiction (5.11), we also must have $\bar{v}_{cd}(\hat{c}_+(d_0), d_0) < 0$. So we are now reduced to the contradiction assumption of item 1, we can apply the argument of that item and get the contradiction, so the claim. □

Remark 5.1. In [34], a similar smooth-fit principle (5.3) is derived a posteriori in the particular case where the state process is a geometric Brownian motion, so that an explicit smooth solution can be obtained, and then shown to be the equal to the value function by a verification approach. In the general diffusion case for demand and when the cost function is quadratic, we prove directly the smooth-fit principle (5.3) by a viscosity solutions approach.

5.2 Characterization of the optimal boundaries

Proposition 5.1 can be used to add other necessary optimality conditions to (5.1): indeed, by (4.40), the relation (5.3) yields

$$A'(c)\psi'(d) + B'(c)\varphi'(d) + \hat{V}_{cd}(c,d) \;=\; 0, \quad \forall\,(c,d) \in \partial\mathcal{C} \tag{5.25}$$

We want to use the optimality conditions (5.1) and (5.25) to characterize the optimal boundaries $\partial\mathcal{C}^{\pm}$. First, we rewrite such conditions. (The proofs of the next two propositions follow the line of [4] and also, in some parts, of [34].)

Proposition 5.2. *Let $c \in \mathbb{R}$ and let $d_+, d_- \in \mathcal{O}$ be such that $(c, d_-) \in \partial^-\mathcal{C}$, $(c, d_+) \in \partial^+\mathcal{C}$. Then*

$$\begin{cases} \displaystyle\int_{d_-}^{d_+} \psi(\xi) g_c(c,\xi) m'(\xi)\mathrm{d}\xi \;+\; q_0^- \frac{\psi'(d_-)}{S'(d_-)} \;+\; q_0^+ \frac{\psi'(d_+)}{S'(d_+)} \;=\; 0, \\[2ex] \displaystyle\int_{d_-}^{d_+} \varphi(\xi) g_c(c,\xi) m'(\xi)\mathrm{d}\xi \;+\; q_0^- \frac{\varphi'(d_-)}{S'(d_-)} \;+\; q_0^+ \frac{\varphi'(d_+)}{S'(d_+)} \;=\; 0. \end{cases} \tag{5.26}$$

Proof. Let $c, d_\pm$ be as in the statement. The conditions (5.1) computed respectively at (c, d_+) and (c, d_-) yield

$$\begin{cases} A'(c)\psi(d_+) + B'(c)\varphi(d_+) + \hat{V}_c(c,d_+) \;=\; -q_0^+, \\ A'(c)\psi(d_-) + B'(c)\varphi(d_-) + \hat{V}_c(c,d_-) \;=\; q_0^-, \end{cases}$$

from which we get

$$\begin{cases} A'(c) \;=\; \frac{\varphi(d_-)(-\hat{V}_c(c,d_+) - q_0^+) - \varphi(d_+)(q_0^- - \hat{V}_c(c,d_-))}{\psi(d_+)\varphi(d_-) - \varphi(d_+)\psi(d_-)}, \\[2ex] B'(c) \;=\; \frac{\psi(d_+)(q_0^- - \hat{V}_c(c,d_-)) - \psi(d_-)(-q_0^+ - \hat{V}_c(c,d_+))}{\psi(d_+)\varphi(d_-) - \varphi(d_+)\psi(d_-)}. \end{cases} \tag{5.27}$$

By Theorem 4.1

$$v_c(c,d) \;=\; A'(c)\psi(d) + B'(c)\varphi(d) + \hat{V}_c(c,d), \quad \forall d \in [d_-, d_+]. \tag{5.28}$$

So, plugging (5.27) into (5.28), we get

$$v_c(c,d) \;=\; \frac{\tilde{\varphi}(d)}{\tilde{\varphi}(d_-)}(q_0^- - \hat{V}_c(c,d_-)) + \frac{\tilde{\psi}(d)}{\tilde{\psi}(d_+)}(-q_0^+ - \hat{V}_c(c,d_+)) + \hat{V}_c(c,d), \;\; \forall d \in [d_-, d_+], \tag{5.29}$$

where

$$\tilde{\varphi}(d) \;:=\; \varphi(d) - \frac{\varphi(d_+)}{\psi(d_+)}\psi(d), \qquad \tilde{\psi}(d) \;:=\; \psi(d) - \frac{\psi(d_-)}{\varphi(d_-)}\varphi(d). \tag{5.30}$$

Hence

$$v_{cd}(c,d) \;=\; \frac{\tilde{\varphi}'(d)}{\tilde{\varphi}(d_-)}(q_0^- - \hat{V}_c(c,d_-)) + \frac{\tilde{\psi}'(d)}{\tilde{\psi}(d_+)}(-q_0^+ - \hat{V}_c(c,d_+)) + \hat{V}_{cd}(c,d), \; \forall d \in [d_-, d_+]. \tag{5.31}$$

Now (5.25) yields $v_{cd}(c, d_-) = v_{cd}(c, d_+) = 0$. Imposing these conditions into (5.31), we get

$$\begin{cases} q_0^- - \hat{V}_c(c, d_-) &= \frac{-\hat{V}_{cd}(c,d_-)\tilde{\psi}'(d_+)\tilde{\varphi}(d_-)+\hat{V}_{cd}(c,d_+)\tilde{\psi}'(d_-)\tilde{\varphi}(d_-)}{\tilde{\varphi}'(d_-)\tilde{\psi}'(d_+)-\tilde{\psi}'(d_-)\tilde{\varphi}'(d_+)}, \\ -q_0^+ - \hat{V}_c(c, d_+) &= \frac{-\hat{V}_{cd}(c,d_+)\tilde{\varphi}'(d_-)\tilde{\psi}(d_+)+\hat{V}_{cd}(c,d_-)\tilde{\varphi}'(d_+)\tilde{\psi}(d_+)}{\tilde{\varphi}'(d_-)\tilde{\psi}'(d_+)-\tilde{\psi}'(d_-)\tilde{\varphi}'(d_+)}. \end{cases} \tag{5.32}$$

Simple computations yield

$$\begin{aligned}
\tilde{\varphi}'(d_-)\tilde{\psi}'(d_+) - \tilde{\psi}'(d_-)\tilde{\varphi}'(d_+) &= (\varphi'(d_-)\psi'(d_+) - \varphi'(d_+)\psi'(d_-))(\varphi(d_-)\psi(d_+) - \varphi(d_+)\psi(d_-)), \\
\tilde{\psi}'(d_+)\tilde{\varphi}(d_-) &= \frac{(\psi'(d_+)\varphi(d_-) - \psi(d_-)\varphi'(d_+))(\varphi(d_-)\psi(d_+) - \varphi(d_+)\psi(d_-))}{\psi(d_+)\varphi(d_-)}, \\
\tilde{\psi}'(d_-)\tilde{\varphi}(d_-) &= \frac{(\psi'(d_-)\varphi(d_-) - \psi(d_-)\varphi'(d_-))(\varphi(d_-)\psi(d_+) - \varphi(d_+)\psi(d_-))}{\psi(d_+)\varphi(d_-)}, \\
\tilde{\varphi}'(d_-)\tilde{\psi}(d_+) &= \frac{(\varphi'(d_-)\psi(d_+) - \varphi(d_+)\psi'(d_-))(\varphi(d_-)\psi(d_+) - \varphi(d_+)\psi(d_-))}{\psi(d_+)\varphi(d_-)}, \\
\tilde{\varphi}'(d_+)\tilde{\psi}(d_+) &= \frac{(\varphi'(d_+)\psi(d_+) - \varphi(d_+)\psi'(d_+))(\varphi(d_-)\psi(d_+) - \varphi(d_+)\psi(d_-))}{\psi(d_+)\varphi(d_-)}.
\end{aligned}$$

Plugging these expressions into (5.32) we get

$$\begin{cases} q_0^- - \hat{V}_c(c, d_-) &= \frac{-\hat{V}_{cd}(c,d_-)(\psi'(d_+)\varphi(d_-)-\psi(d_-)\varphi'(d_+))+\hat{V}_{cd}(c,d_+)(\psi'(d_-)\varphi(d_-)-\psi(d_-)\varphi'(d_-)}{\varphi'(d_-)\psi'(d_+)-\psi'(d_-)\varphi'(d_+)}, \\ -q_0^+ - \hat{V}_c(c, d_+) &= \frac{-\hat{V}_{cd}(c,d_+)(\varphi'(d_-)\psi(d_+)-\varphi(d_+)\psi'(d_-))+\hat{V}_{cd}(c,-)(\varphi'(d_+)\psi(d_+)-\varphi(d_+)\psi'(d_+))}{\varphi'(d_-)\psi'(d_+)-\psi'(d_-)\varphi'(d_+)}. \end{cases} \tag{5.33}$$

Using the representations (3.10)-(3.11) in (5.33), we get after long computations

$$\begin{aligned}
-q_0^+(\varphi'(d_-)\psi'(d_+) - \psi'(d_-)\varphi'(d_+)) &= \varphi'(d_-)S'(d_+)\int_{d_-}^{d_+} \psi(\xi)g_c(c,\xi)m'(\xi)\mathrm{d}\xi \\
&\quad -\psi'(d_-)S'(d_+)\int_{d_-}^{d_+} \varphi(\xi)g_c(c,\xi)m'(\xi)\mathrm{d}\xi,
\end{aligned}$$

$$\begin{aligned}
q_0^-(\varphi'(d_-)\psi'(d_+) - \psi'(d_-)\varphi'(d_+)) &= \varphi'(d_+)S'(d_-)\int_{d_-}^{d_+} \psi(\xi)g_c(c,\xi)m'(\xi)\mathrm{d}\xi \\
&\quad -\psi'(d_+)S'(d_-)\int_{d_-}^{d_+} \varphi(\xi)g_c(c,\xi)m'(\xi)\mathrm{d}\xi,
\end{aligned}$$

from which we finally see that the couple $(d_-, d_+) \in \mathcal{O} \times \mathcal{O}$ satisfies (5.26). □

Let us denote

$$\underline{c}_{+,g} := \inf_{\mathcal{O}} \hat{c}_{+,g}, \quad \underline{c}_{-,g} := \inf_{\mathcal{O}} \hat{c}_{-,g}, \quad \bar{c}_{+,g} := \sup_{\mathcal{O}} \hat{c}_{+,g}, \quad \bar{c}_{-,g} := \sup_{\mathcal{O}} \hat{c}_{-,g}.$$

For all $c \in \mathbb{R}$ denote

$$d_+^*(c) := \inf\{\xi \in \mathcal{O} \mid g_c(c,\xi) < -\rho q_0^+\}, \quad d_-^*(c) := \sup\{\xi \in \mathcal{O} \mid g_c(c,\xi) > \rho q_0^-\}.$$

with the convention $\sup\emptyset = d_{\min}$, $\inf\emptyset = d_{\max}$. Then clearly we have $d_+^*(c) < d_-^*(c)$ for every $c \in \mathbb{R}$, and $d_+^*(c), d_-^*(c) \in \mathcal{O}$ if and only if $c \in (\underline{c}_{-,g}, \bar{c}_{+,g})$.

Proposition 5.3. *Let $c \in \mathbb{R}$ and let $-\beta_0$ be strictly decreasing (so that $g_c(c,\cdot) = -\beta_0(\cdot)$ is strictly decreasing for every $c \in \mathbb{R}$). The couple of equations*

$$\begin{cases} \displaystyle\int_x^y \psi(\xi) g_c(c,\xi) m'(\xi) \mathrm{d}\xi \; + \; q_0^- \frac{\psi'(x)}{S'(x)} \; + \; q_0^+ \frac{\psi'(y)}{S'(y)} \;\; = \;\; 0 \\ \\ \displaystyle\int_x^y \varphi(\xi) g_c(c,\xi) m'(\xi) \mathrm{d}\xi \; + \; q_0^- \frac{\varphi'(x)}{S'(x)} \; + \; q_0^+ \frac{\varphi'(y)}{S'(y)} \;\; = \;\; 0. \end{cases} \tag{5.34}$$

admits a solution $(x^*(c), y^*(c))$ with $y^*(c) > x^*(c)$ if and only if $c \in (\underline{c}_{-,g}, \bar{c}_{+,g})$ (note that the case $\underline{c}_{-,g} > \bar{c}_{+,g}$ may occur, and in this case this interval is considered as empty). If this is the case, i.e. $c \in (\underline{c}_{-,g}, \bar{c}_{+,g})$, then the solution is unique and belongs to $(d_{\min}, d_-^*(c)) \times (d_+^*(c), d_{\max})$.

Moreover x^*, y^* are continuously differentiable in the interval $(\underline{c}_{-,g}, \bar{c}_{+,g})$ and have strictly positive derivatives.

Proof. Fix $c \in \mathbb{R}$ and consider the functions in the couple of variables $(x,y) \in \mathcal{O} \times \mathcal{O}$

$$L_1(x,y;c) \;\; := \;\; \int_x^y \psi(\xi) g_c(c,\xi) m'(\xi) \mathrm{d}\xi \; + \; q_0^+ \frac{\psi'(y)}{S'(y)} \; + \; q_0^- \frac{\psi'(x)}{S'(x)}, \tag{5.35}$$

$$L_2(x,y;c) \;\; := \;\; \int_x^y \varphi(\xi) g_c(c,\xi) m'(\xi) \mathrm{d}\xi \; + \; q_0^+ \frac{\varphi'(y)}{S'(y)} \; + \; q_0^- \frac{\varphi'(x)}{S'(x)}. \tag{5.36}$$

The solvability of our system of equations corresponds then to the solvability of $L_1(x,y;c) = 0$, $L_2(x,y;c) = 0$ in $\mathcal{O} \times \mathcal{O}$ with $x < y$. Using the representations (see, e.g., [8, Ch. II])

$$\frac{\psi'(\cdot)}{S'(\cdot)} \;\; = \;\; \rho \int_{d_{\min}}^{\cdot} \psi(\xi) m'(\xi) \mathrm{d}\xi, \qquad \frac{\varphi'(\cdot)}{S'(\cdot)} \;\; = \;\; -\rho \int_{\cdot}^{d_{\max}} \varphi(\xi) m'(\xi) \mathrm{d}\xi, \tag{5.37}$$

L_1, L_2 can be rewritten as

$$L_1(x,y;c) \;\; = \;\; \int_x^y \psi(\xi)(g_c(c,\xi) + \rho q_0^+) m'(\xi) \mathrm{d}\xi \; + \; (q_0^+ + q_0^-) \frac{\psi'(x)}{S'(x)},$$

$$L_2(x,y;c) \;\; = \;\; \int_x^y \varphi(\xi)(g_c(c,\xi) - \rho q_0^-) m'(\xi) \mathrm{d}\xi \; + \; (q_0^+ + q_0^-) \frac{\varphi'(y)}{S'(y)},$$

or equivalently as

$$L_1(x,y;c) \;\; = \;\; \int_x^y \psi(\xi) g_c(c,\xi) m'(\xi) \mathrm{d}\xi \; + \; \rho q_0^+ \int_{d_{\min}}^y \psi(\xi) m'(\xi) \mathrm{d}\xi \; + \; \rho q_0^- \int_{d_{\min}}^x \psi(\xi) m'(\xi) \mathrm{d}\xi,$$

$$L_2(x,y;c) \;\; = \;\; \int_x^y \varphi(\xi) g_c(c,\xi) m'(\xi) \mathrm{d}\xi \;\; - \;\; \rho q_0^+ \int_y^{d_{\max}} \varphi(\xi) m'(\xi) \mathrm{d}\xi \; - \; \rho q_0^- \int_x^{d_{\max}} \varphi(\xi) m'(\xi) \mathrm{d}\xi,$$

and the partial derivatives of L_1, L_2 with respect to x, y are

$$\frac{\partial L_1}{\partial x}(x,y;c) \;\; = \;\; -\psi(x)(g_c(c,x) - \rho q_0^-) m'(x), \qquad \frac{\partial L_1}{\partial y}(x,y;c) \;\; = \;\; \psi(y)(g_c(c,y) + \rho q_0^+) m'(y),$$

$$\frac{\partial L_2}{\partial x}(x,y;c) \;=\; -\varphi(x)(g_c(c,x)-\rho q_0^-)m'(x), \qquad \frac{\partial L_2}{\partial y}(x,y;c) \;=\; \varphi(y)(g_c(c,y)+\rho q_0^+)m'(y).$$

Let us study the solvability of $L_1(x,\cdot;c)=0$ for given $x\in\mathcal{O}$. First of all we notice that $L_1(x,x;c)>0$ as $\psi'>0$, $S'>0$. Taking into account that $g_c(c,\cdot)$ is strictly decreasing and continuous, we see that the sign of $\frac{\partial L_1}{\partial y}(x,\cdot;c)$ is strictly positive in $(x,d_+^*(c))$ and strictly negative in $(d_+^*(c),d_{\max})$. Combined with the fact that $L_1(x,x;c)>0$, this shows that there is at most one point $y^*(x;c)\in(x,d_{\max})$ solution to $L_1(x,\cdot;c)=0$ and that $y^*(x;c)$ (if exists) must belong to $(d_+^*(c),d_{\max})$. Now we distinguish two cases.

- If $c\geq \bar{c}_{+,g}$, then $g_c(c,\cdot)+\rho q_0^+\geq 0$ in $\mathcal{O}$. So the solution does not exist in this case.
- If $c<\bar{c}_{+,g}$, take $\hat{y}(c)>d^*(c)$ such that $L_1(x,\hat{y}(c);c)>0$ (such $\hat{y}(c)$ exists by continuity), and observe that since $g_c(c,\cdot)$ is (strictly) decreasing, using (5.37), one has for every $y\geq\hat{y}(c)$

  $$\int_{\hat{y}(c)}^{y}\psi(\xi)m'(\xi)(g_c(c,\xi)+\rho q_0^+)\mathrm{d}\xi \;\leq\; \frac{g_c(c,\hat{y})+\rho q_0^+}{\rho}\left(\frac{\psi'(y)}{S'(y)}-\frac{\psi'(\hat{y})}{S'(\hat{y})}\right),$$

  therefore

  $$L_1(x,y;c) \;\leq\; L_1(x,\hat{y}(c);c) \;+\; \frac{g_c(c,\hat{y}(c))+\rho q_0^+}{\rho}\left(\frac{\psi'(y)}{S'(y)}-\frac{\psi'(\hat{y})}{S'(\hat{y})}\right). \tag{5.38}$$

  Now we notice that there exists $M_c>0$ such that $L_1(x,\hat{y}(c);c)\leq M_c$ for every $x\leq\hat{y}(c)$. Indeed, $\int_{d_{\min}}^{\hat{y}(c)}\psi(\xi)g_c(c,\xi)m'(\xi)\mathrm{d}\xi$ is finite because of the finiteness of $\hat{V}_c$ and taking into account (3.11); $\int_{d_{\min}}^{\hat{y}(c)}\psi(\xi)m'(\xi)\mathrm{d}\xi$ is finite because of (5.37); $\psi'(x)/S'(x)$ is bounded in $(d_{\min},\hat{y}(c)]$ because of (3.9). Now, since $g_c(c,\hat{y}(c))+\rho q_0^+<0$ and since by (3.9) we have $\psi'(y)/S'(y)\to\infty$ as $y\to d_{\max}$, we see that the solution $y^*(x;c)$ to $L_1(x,\cdot;c)=0$ exists in the interval $(\hat{y}(c),d_{\max}-\varepsilon_{M_c}]$ for some $\varepsilon_{M_c}>0$, hence in the interval $(d_+^*(c),d_{\max}-\varepsilon_{M_c}]$, for every $x\leq d_-^*(c)$.

Hence we have shown that, given $x\in\mathcal{O}$, there exists a unique solution $y^*(x;c)$ to $L_1(x,\cdot;c)=0$ if and only if $c<\bar{c}_{+,g}$, and it belongs to the interval $(d_+^*(c),d_{\max}-\varepsilon_{M_c}]$. Morever, Implicit Function Theorem ensures that $y^*(\cdot;c)$ is continuously differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}x}y^*(x;c) \;=\; -\frac{\frac{\partial L_1}{\partial x}(x,y^*(x;c))}{\frac{\partial L_1}{\partial y}(x,y^*(x;c))} \;=\; \frac{\psi(x)m'(x)(g_c(c,x)-\rho q_0^-)}{\psi(y^*(x))m'(y^*(x))(g_c(c,y^*(x))+\rho q_0^+)}. \tag{5.39}$$

Now consider the equation $L_2(x,y^*(x;c);c)=0$. We are going to show existence and uniqueness of solutions to such equation in $\mathcal{O}$. This will complete the proof of existence and uniqueness of solutions for (5.34), as, from what we have said before, $x^*(c)$ solves the latter equation if and only if $(x^*(c),y^*(x^*(c);c))$ solves (5.34). We observe that:

- If $c\leq\underline{c}_{-,g}$, then $g_c(c,\cdot)-\rho q_0^-\leq 0$ in $\mathcal{O}$; so, since $\varphi'(\cdot)/S'(\cdot)<0$ we have $L_2(\cdot,y^*(\cdot;c))<0$ in $\mathcal{O}$ and the solution does not exist.
- If $c>\underline{c}_{-,g}$, then we have the following facts:
  1. $L_2(\cdot;y^*(\cdot;c))<0$ in $(d_-^*(c),d_{\max})$, as $g_c(c,\cdot)-\rho q_0^-\leq 0$ therein and $\varphi'(\cdot)/S'(\cdot)<0$.

2. Using (5.39) we compute

$$\frac{\mathrm{d}}{\mathrm{d}x}L_2(x,y^*(x;c)) \quad = \quad \frac{\psi(x)\varphi(y^*(x;c))-\psi(y^*(x;c))\varphi(x)}{\psi(y^*(x;c)}m'(x)(g_c(c,x)-\rho q_0^-).$$

So taking into account that $y^*(x;c)>x$, the strict (opposite) monotonicity of φ,ψ, and that $g(c,\cdot)-\rho q_0^- > 0$ in $(d_{\min},d_-^*(c))$, we see that $\frac{\mathrm{d}}{\mathrm{d}x}L_2(x,y^*(x;c)) < 0$ for $x\in(d_{\min},d_-^*(c))$.

3. Arguing as in proving (5.38), we can prove that there exists $\hat{x}\in(d_{\min},d_-^*(c))$ such that $L_2(\hat{x},y^*(\hat{x};c))<0$ and

$$\begin{aligned} L_2(x,y^*(x;c)) \quad \geq \quad & \int_{\hat{x}}^{y^*(x;c)} \varphi(\xi)(g_c(c,\xi)-\rho q_0^-)m'(\xi)\mathrm{d}\xi \\ & -\frac{g_c(c,\hat{x})-\rho q_0^-}{\rho}\left(\frac{\varphi'(x)}{S'(x)}-\frac{\varphi'(\hat{x})}{S'(\hat{x})}\right). \end{aligned}$$

Since $y^*(x;c)\in(d_+^*(c),d_{\max}-\varepsilon_{M_c}]$ for every $x\in(d_{\min},d_-^*(c)]$, setting

$$K_0 \quad := \quad \int_{\hat{x}}^{d_{\max}-\varepsilon_{M_c}} \varphi(\xi)(g_c(c,\xi)-\rho q_0^-)m'(\xi)\mathrm{d}\xi,$$

the latter inequality yields

$$L_2(x,y^*(x;c)) \quad \geq \quad K_0 - \frac{g_c(c,\hat{x})-\rho q_0^-}{\rho}\left(\frac{\varphi'(x)}{S'(x)}-\frac{\varphi'(\hat{x})}{S'(\hat{x})}\right).$$

Now, since $\frac{\varphi'(x)}{S'(x)}\to-\infty$ as $x\to d_{\min}$ due to (3.9), and since $g_c(c,\hat{x})-\rho q_0^- > 0$, we see that $L_2(x,y^*(x;c))\to\infty$ as $x\to d_{\min}$.

Combining these three fact we deduce that there exists a unique solution to the equation $L_2(\cdot;y^*(\cdot;c))=0$ and that it belongs to the interval $(d_{\min},d_-^*(c))$. □

Let us show now the last part of the claim. Consider c as a variable in L_1,L_2 and consider the matrix

$$\mathcal{M}(x^*(c),y^*(c);c) \quad = \quad \begin{pmatrix} \frac{\partial L_1}{\partial x}(x^*(c),y^*(c);c) & \frac{\partial L_1}{\partial y}(x^*(c),y^*(c);c) \\ \frac{\partial L_2}{\partial x}(x^*(c),y^*(c);c) & \frac{\partial L_2}{\partial y}(x^*(c),y^*(c);c) \end{pmatrix}.$$

Taking into account that $x^*(c)<d_-^*(c)$, $y^*(c)>d_-^*(c)$, and that ψ,φ are respectively strictly increasing and strictly decreasing, we see that the $\mathcal{M}^*(x^*(c),y^*(c);c)$ is actually non singular. More precisely $M:=\det(\mathcal{M}(x^*(c),y^*(c);c))<0$ and

$$\mathcal{M}(x^*(c),y^*(c);c)^{-1} \quad = \quad \frac{1}{M}\begin{pmatrix} \frac{\partial L_2}{\partial y}(x^*(c),y^*(c);c) & -\frac{\partial L_1}{\partial y}(x^*(c),y^*(c);c) \\ -\frac{\partial L_2}{\partial x}(x^*(c),y^*(c);c) & \frac{\partial L_1}{\partial x}(x^*(c),y^*(c);c) \end{pmatrix}. \tag{5.40}$$

So, since $L_1(x^*(c),y^*(c);c)=0$, $L_1(x^*(c),y^*(c);c)=0$, we can apply Implicit Function Theorem which yields

$$\frac{\mathrm{d}}{\mathrm{d}c}\begin{pmatrix} x^*(c) \\ y^*(c) \end{pmatrix} \quad = \quad -\mathcal{M}(x^*(c),y^*(c);c)^{-1}\begin{pmatrix} \frac{\partial L_1}{\partial c}(x^*(c),y^*(c);c) \\ \frac{\partial L_2}{\partial c}(x^*(c),y^*(c);c) \end{pmatrix}. \tag{5.41}$$

Since $g_{cc} = 1$, we have

$$\frac{\partial L_1}{\partial c}(x^*(c), y^*(c); c) = \int_{x^*(c)}^{y^*(c)} \psi(\xi)m'(\xi)\mathrm{d}\xi, \qquad \frac{\partial L_2}{\partial c}(x^*(c), y^*(c); c) = \int_{x^*(c)}^{y^*(c)} \varphi(\xi)m'(\xi)\mathrm{d}\xi.$$

So, from (5.41)-(5.40) we get

$$\frac{\mathrm{d}}{\mathrm{d}c}x^*(c) = -\frac{1}{M}(g_c(c, y^*(c)) + \rho q_0^+)m'(y^*(c)) \int_{x^*(c)}^{y^*(c)} (\varphi(y^*(c))\psi(\xi) - \psi(y^*(c))\varphi(\xi))m'(\xi)\mathrm{d}\xi$$

$$\frac{\mathrm{d}}{\mathrm{d}c}y^*(c) = -\frac{1}{M}(g_c(c, x^*(c)) - \rho q_0^-)m'(x^*(c)) \int_{x^*(c)}^{y^*(c)} (\varphi(x^*(c))\psi(\xi) - \psi(x^*(c))\varphi(\xi))m'(\xi)\mathrm{d}\xi.$$

Now, notice that

$$M < 0, \qquad g_c(c, y^*(c)) + \rho q_0^+ < 0, \qquad g_c(c, x^*(c)) - \rho q_0^- > 0,$$

and that the functions

$$q(\xi) := \varphi(y^*(c))\psi(\xi) - \psi(y^*(c))\varphi(\xi), \qquad p(\xi) := \varphi(x^*(c))\psi(\xi) - \psi(x^*(c))\varphi(\xi),$$

are both strictly increasing and verify, respectively $q(y^*(c)) = 0$ and $p(x^*(c)) = 0$. So we conclude from (5.41). □

We are now ready to characterize the optimal boundaries.

Theorem 5.1. *Let $-\beta_0$ be strictly decreasing. We have $\underline{c}_- = \underline{c}_{-,g}$, $\bar{c}_+ = \bar{c}_{+,g}$ and the optimal boundaries $\partial^\pm\mathcal{C}$ are characterized piecewise as follows. (Note that some of the three regions below where we split the characterization may be empty.)*

1. *In the region $(\underline{c}_{-,g}, \bar{c}_{+,g}) \times \mathcal{O}$, the optimal boundaries $\partial^\pm\mathcal{C}$ are identified by the functions $\hat{d}_\pm$ which are characterized as follows: given $c \in (\underline{c}_{-,g}, \bar{c}_{+,g})$ the couple $(\hat{d}_-(c), \hat{d}_+(c)) \in \mathcal{O} \times \mathcal{O}$ is the unique solution of the system of equations* (5.34) *provided by Proposition 5.3.*

2. *In the region $(-\infty, \underline{c}_{-,g}] \times \mathcal{O}$ only $\partial^+\mathcal{C}$ (at most) exists and is identified in terms of the function $\hat{c}_+$ (note that Corollary 4.1 ensures $\hat{c}_+ > -\infty$)), which is explicitly given by*

$$\hat{c}_+(d) = \rho\left[\beta(d) - \frac{\psi(d)}{\psi'(d)}\beta'(d) - q_0^+\right], \quad d \leq \lim_{c \downarrow \underline{c}_{-,g}} \hat{d}_+(c), \quad d \in \mathcal{O}.$$

   (For the definition of $\lim_{c\downarrow\bar{c}_{-,g}} \hat{d}_+(c)$ when $(\underline{c}_{-,g}, \bar{c}_{+,g})$ is empty, recall that $\hat{d}_+(c) \equiv d_{\max}$ for $c \geq \bar{c}_+$.)

3. *In the region $[\bar{c}_{+,g}, \infty) \times \mathcal{O}$ only $\partial^-\mathcal{C}$ (at most) exists and is identified in terms of the function $\hat{c}_-$ (note that Corollary 4.1 ensures $\hat{c}_- < \infty$)), which is explicitly given by*

$$\hat{c}_-(d) = \rho\left[\beta(d) - \frac{\varphi(d)}{\varphi'(d)}\beta'(d) + q_0^-\right], \quad d \geq \lim_{c \uparrow \bar{c}_{+,g}} \hat{d}_-(c), \quad d \in \mathcal{O}.$$

   (For the definition of $\lim_{c\uparrow\bar{c}_{+,g}} \hat{d}_-(c)$ when $(\underline{c}_{-,g}, \bar{c}_{+,g})$ is empty, recall that $\hat{d}_-(c) \equiv d_{\min}$ for $c \leq \underline{c}_-$.)

Moreover:

(i) The functions $\hat{c}_\pm : \mathcal{O} \to \mathbb{R}$ *are continuous and strictly increasing.*

(ii) $\hat{c}_+$ *and* $\hat{c}_-$ *are of class* C^1 *except, at most, at the points* $\lim_{c\downarrow \underline{c}_{-,g}} \hat{d}_+(c)$ *and* $\lim_{c\uparrow \bar{c}_{+,g}} \hat{d}_-(c)$*, respectively (if they belong to* $\mathcal{O}$*).*

Proof. 1. First of all we notice that, by Proposition 4.4, we have $\underline{c}_{-,g} \leq \underline{c}_-$ and $\bar{c}_{+,g} \geq \bar{c}_+$. In the interval $(\underline{c}_-, \bar{c}_+)$, we have that the couple $(\hat{d}_-(c), \hat{d}_+(c))$ belongs to $\mathcal{O} \times \mathcal{O}$, and, by Propositions 5.2 and 5.3, it can be identified as the unique solution of the system of equations (5.34). This shows claim 1, once we prove the claim $\underline{c}_{-,g} = \underline{c}_-$ and $\bar{c}_{+,g} = \bar{c}_+$, which is what we are going to prove now.

Assume by contradiction that $(\underline{c}_{-,g}, \underline{c}_-]$ is nonempty. Then, for all $c \in (\underline{c}_{-,g}, \underline{c}_-]$ we should have a unique solution $(d_-(c), d_+(c)) \in \mathcal{O} \times \mathcal{O}$ to (5.34) as provided by Proposition 5.3. By the monotonicity claim of Proposition 5.3, such a solution should be such that $d_{\min} < d_-(c) < \lim_{\zeta\downarrow \underline{c}_-} \hat{d}_-(\zeta) =: d_0$. Now if $d_0 > d_{\min}$, then, by definition of $\underline{c}_-$ we would have $\hat{c}_- \equiv \underline{c}_-$ in $(d_{\min}, d_0)$ and we would have, by Proposition 5.2, more than one solution to (5.34) at the level $\underline{c}_-$. But this contradicts Proposition 5.3. Therefore it should be $d_0 = d_{\min}$, but this would be a contradiction to $d_{\min} < d_-(c) < d_0$. Hence, it remains proved that $\underline{c}_- = \underline{c}_{-,g}$. The same argument applies to $\bar{c}_+$ and so the claim is proved.

2. The fact that only $\partial^+\mathcal{C}$ (at most) exists in the region $(-\infty, \underline{c}_{-,g}]$ is due to the definition of $\underline{c}_-$, to the equality $\underline{c}_{-,g} = \underline{c}_-$ and to the fact that, as shown in item 1, $\lim_{\zeta\downarrow\underline{c}_-} \hat{d}_-(\zeta) = d_{\min}$. Then, due to Theorem 4.1, we have $B(c) = 0$ for all $c \leq \underline{c}_{-,g}$. Hence, the optimality conditions (5.1) and (5.25) written at the points $(\hat{c}_+(d), d) \in \partial^+\mathcal{C}$ with $d \in (d_{\min}, \lim_{c\downarrow\underline{c}_{-,g}} \hat{d}_+(c)]$ (notice that, due to Corollary 4.1, we actually have $\hat{c}_+ : \mathcal{O} \to \mathbb{R}$) yield

$$\begin{cases} A'(\hat{c}_+(d))\psi(d) + \frac{1}{\rho}\hat{c}_+(d) - \beta(d) &= -q_0^+, \\ \\ A'(\hat{c}_+(d))\psi'(d) - \beta'(d) &= 0. \end{cases} \tag{5.42}$$

Multiplying the second equation in (5.42) by ψ/ψ' and subtracting it to the first one, we get (5.45).

3. The same argument of item 2 applies symmetrically.

Let us now show items (i) and (ii).

(i) We show the claim for $\hat{c}_+$, the proof of the claim regarding $\hat{c}_-$ is analogous.

Since $\hat{d}_+$ is strictly increasing and continuous in the interval $(\underline{c}_{-,g}, \bar{c}_{+,g})$ (when this is not empty), we see that $\hat{c}_+$ is the inverse of $\hat{d}_+$ in the interval $(\lim_{c\downarrow\underline{c}} \hat{d}_+(c), d_{\max})$ (when this is, correspondingly, nonempty) and is strictly increasing and continuous therein. So we must now show that $\hat{c}_+$ is strictly increasing and continuous in the interval $(d_{\min}, \lim_{c\downarrow\underline{c}} \hat{d}_+(c)]$ (when this is nonempty). Assume by contradiction that there exists a nonempty interval $(a, b) \subset (d_{\min}, \lim_{c\downarrow\underline{c}} \hat{d}_+(c)]$ where $\hat{c}_+ \equiv c_0$. Then from the first equality in (5.42) we should have

$$\beta(d) \quad = \quad A'(c_0)\psi(d) + \frac{1}{\rho}c_0 + q_0^+, \qquad d \in (a, b).$$

Since ψ solves $\mathcal{L}\psi = 0$, we then have that $\mathcal{L}\beta \equiv c_0 + \rho q_0^+$ in (a,b). On the other hand, from (5.5), we see that it must be also $\mathcal{L}\beta = \beta_0$, so we should conclude that β_0 is constant in (a,b), contradictiong the hypothesis. So, it has been proved that $\hat{c}_+$ is strictly increasing.

Now we show that $\hat{c}_+$ is continuous. Indeed it is continuous in the interval $(d_{\min}, \lim_{c\downarrow \underline{c}} \hat{d}_+(c)]$ due to item 2, and in the interval $(\lim_{c\downarrow \underline{c}} \hat{d}_+(c), d_{\max})$, due to item 1. It remains to prove that $\hat{c}_+$ is continuous at $\lim_{c\downarrow \underline{c}} \hat{d}_+(c)$ (when it belongs to $\mathcal{O}$). This comes just from the fact that $\hat{c}_+$ is right-continuous in general and, as we have seen just now, it is left-continuous at $\lim_{c\downarrow \underline{c}} \hat{d}_+(c)$.

(ii) It follows from the previuos claims and from Proposition 5.3. □

We notice that $\underline{c}_{-,g}$, $\bar{c}_{+,g}$ are explicit. So Theorem 5.1 actually provides a way to find, up to the (possibly numerical) solution of the system of equations (5.34) for every $c \in (\underline{c}_{-,g}, \bar{c}_{+,g})$, when this interval is not empty, the optimal boundaries $\partial^{\pm}\mathcal{C}$. Then the functions A, B individuating the value function in the continuation region can be retrieved by Theorem 4.1:

- If $(\underline{c}_{-,g}, \bar{c}_{+,g}) \neq \emptyset$, then A, B can be computed in the interval $(\underline{c}_{-,g}, \bar{c}_{+,g})$ by integrating (5.27) with boundary conditions $A(\bar{c}_{+,g}) = 0$ and $B(\underline{c}_{-,g}) = 0$, and, respectively in the intervals $(\underline{c}_+, \underline{c}_{-,g}]$ and $[\bar{c}_+, \bar{c}_{+,g})$ (when they are nonempty), by the equalities

$$\begin{aligned} A(c) &= [\lim_{c\downarrow \underline{c}_{-,g}} A(c)] - \int_c^{\underline{c}_{-,g}} \frac{\beta'(\hat{d}_+(\xi))}{\psi'(\hat{d}_+(\xi))} \mathrm{d}\xi, \qquad c \in (\underline{c}_+, \underline{c}_{-,g}]; \\ B(c) &= [\lim_{c\uparrow \bar{c}_{+,g}} B(c)] + \int_{\bar{c}_{+,g}}^c \frac{\beta'(\hat{d}_+(\xi))}{\varphi'(\hat{d}_+(\xi))} \mathrm{d}\xi, \qquad c \in [\bar{c}_{+,g}, \bar{c}_-). \end{aligned}$$

- If $(\underline{c}_{-,g}, \bar{c}_{+,g}) = \emptyset$, then

$$A(c) = -\int_c^{\bar{c}_{+,g}} \frac{\beta'(\hat{d}_+(\xi))}{\psi'(\hat{d}_+(\xi))} \mathrm{d}\xi, \qquad c \in (\underline{c}_+, \bar{c}_{+,g}), \tag{5.43}$$

$$B(c) = \int_{\underline{c}_{-,g}}^c \frac{\beta'(\hat{d}_-(\xi))}{\varphi'(\hat{d}_+(\xi))} \mathrm{d}\xi, \qquad c \in (\underline{c}_{-,g}, \bar{c}_-). \tag{5.44}$$

Then $z_{\pm}$ can be obtained by (4.35).

5.3 Quadratic cost and irreversibility

In this subsection we consider we further particularize to the irreversible investment case. Even if it is, rigorously speaking, out of our setting, nonetheless it can be formally seen as corresponding to take $q_0^- = \infty$. The upper boundary in this case is clearly $\hat{c}_- \equiv \infty$, or, in other terms, it disappears. Hence, from Theorem 5.1, we immediately get the following.

Corollary 5.2. *Let $q_0^- = \infty$, and let the assumptions of Theorem 5.1 hold true. Then the functions $\hat{c}_{\pm}, A, B, z_{\pm}$ of Theorem 4.1 are determined as follows:*

(a) The upper optimal boundary is $\hat{c}_- \equiv \infty$, and lower boundary function $\hat{c}_+$ is explicitly given by

$$\hat{c}_+(d) = \rho \left[\beta(d) - \frac{\psi(d)}{\psi'(d)} \beta'(d) - q_0^+ \right], \quad d \in \mathcal{O}. \tag{5.45}$$

In particular $\hat{c}_+ \in C^1(\mathcal{O}; \mathbb{R})$.

(b) $B \equiv 0$, *and the function* A *is given by*

$$A(c) \;=\; -\int_c^{\bar{c}_{+,g}} \frac{\beta'(\hat{d}_+(\xi))}{\psi'(\hat{d}_+(\xi))}\mathrm{d}\xi, \qquad c \in (\underline{c}_+, \bar{c}_{+,g}),$$

(c) The function z_- *is whatever function (it does not play a role, as* $\hat{c}_- \equiv \infty$ *implies* $\mathcal{A}^- = \emptyset$*), while the function* z_+ *is*

$$z_+(d) \;=\; A(\hat{c}_+(d))\psi(d) + \hat{V}(\hat{c}_+(d), d) + q_0^+\hat{c}_+(d), \qquad d \in \mathcal{O}, \tag{5.46}$$

with $\hat{V}$ *given in* (5.6).

We end this paper by a simple and explicit illustration of our Corollary 5.2 to the case when the demand is modeled as a geometric Brownian motion:

$$\mathrm{d}D_t \;=\; \mu D_t \mathrm{d}t + \sigma D_t \mathrm{d}W_t, \quad \mu \in \mathbb{R},\; \sigma > 0,$$

with initial datum $d > 0$. In this case $\mathcal{O} = (0, \infty)$. Moreover, assume that

$$g(c,d) \;=\; \frac{1}{2}(c-d)^2,$$

and, according to (3.3), assume that

$$\rho \;>\; [2\mu + \sigma^2]^+. \tag{5.47}$$

Then $\hat{V}$ is the quadratic function equal to

$$\hat{V}(c,d) \;=\; \frac{1}{2}\Big(\frac{1}{\rho - 2\mu - \sigma^2}d^2 - \frac{2}{\rho - \mu}dc + \frac{1}{\rho}c^2\Big).$$

The increasing fundamental solution to

$$[\mathcal{L}\phi](d) \;:=\; \rho\phi - \mu d\phi' - \frac{1}{2}\sigma^2 d^2\phi'' \;=\; 0,$$

is given by

$$\psi(d) \;=\; d^m,$$

where m is the positive root of the equation $\rho - \mu m - \frac{1}{2}\sigma^2 m(m-1) = 0$, and explicitly given by

$$m \;=\; -\frac{\mu}{\sigma^2} + \frac{1}{2} + \sqrt{\Big(-\frac{\mu}{\sigma^2} + \frac{1}{2}\Big)^2 + \frac{2\rho}{\sigma^2}}$$

Notice that $m > 2$ by (5.47). From Corollary 5.2, the value function v has the explicit form

$$v(c,d) \;=\; \begin{cases} A(c)d^m + \hat{V}(c,d), & \text{if } c > \hat{c}_+(d), \\ -q_0^+ c + z(d), & \text{if } c \le \hat{c}_+(d). \end{cases}$$

where the functions $A, \hat{c}_+, z$ are

$$\begin{aligned} \hat{c}_+(d) \;&=\; ad - b, \qquad d > 0, \\ A(c) \;&=\; -\frac{a^{m-1}}{m(m-2)}\frac{1}{\rho - \mu}(c+b)^{2-m}, \qquad c > -b, \\ z_+(d) \;&=\; A(ad-b)d^m + \hat{V}(ad-b, d) + q_0^+(\alpha d - b), \qquad d > 0, \end{aligned}$$

with

$$a \;=\; \frac{m-1}{m}\frac{\rho}{\rho - \mu}, \qquad b = \rho q_0^+.$$

A Appendix

Proof of Proposition 3.2. *Existence.* Let $(c,d) \in \mathcal{S}$ and take a sequence $(I^n)_{n\in\mathbb{N}} \subset \mathcal{I}$ s.t. $G(c,d;I^n) \to v(c,d)$. Assume, without loss of generality, that $G(c,d;I^n) \leq v(c,d)+1$ for all $n \geq 0$ and set $\kappa := \min\{q_0^+, q_0^-\} > 0$. Then, taking into account that $g \geq 0$, that $I_{0^-}^{n,+} = I_{0^-}^{n,-} = 0$ for all $n \geq 0$, and integrating by parts, we get

$$
\begin{aligned}
v(c,d)+1 \quad &\geq \quad \kappa\, \mathbb{E} \int_0^\infty e^{-\rho t} \left(\mathrm{d}I_t^{n,+} + \mathrm{d}I_t^{n,-} \right) \\
&= \quad \kappa\, \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(I_t^{n,+} + I_t^{n,-} \right) \mathrm{d}t + [e^{-\rho t}(I_t^{n,+} + I_t^{n,-})]_{0^-}^\infty \right] \\
&\geq \quad \kappa\, \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(I_t^{n,+} + I_t^{n,-} \right) \mathrm{d}t \right].
\end{aligned}
$$

So, the sequence $(I^n)_{n\in\mathbb{N}}$ is bounded in the space $L^1(\Omega \times \mathbb{R}; \mathbb{P} \times e^{-\rho t}\mathrm{d}t)$. Thus, by a theorem of Komlós, there exists a subsequence (relabeled and still denoted by $(I^n)_{n\in\mathbb{N}}$) and a pair of measurable processes $\tilde{I}^+, \tilde{I}^-$ such that the Cesàro sequences of processes

$$
\left(\tilde{I}^{n,\pm} \ := \ \frac{1}{n} \sum_{j=1}^n I^{n,\pm} \right) \subset \mathcal{I} \quad \text{converge} \quad (\mathbb{P} \times e^{-\rho t}\mathrm{d}t) - \text{a.e. to } \tilde{I}^{\pm}. \tag{A.48}
$$

Define $\tilde{I}^n := \tilde{I}^{n,+} - \tilde{I}^{n,-}$. Then, from (A.48), we have the convergence

$$
\tilde{I}^n \quad \longrightarrow \quad \tilde{I} \qquad (\mathbb{P} \times e^{-\rho t}\mathrm{d}t) - \text{a.e.}. \tag{A.49}
$$

By convexity of G w.r.t. the control argument I, we have that also $(\tilde{I}^n)_{n\in\mathbb{N}}$ is a minimizing sequence, i.e. $G(c,d;\tilde{I}^n) \to v(c,d)$. On the other hand, arguing as in Lemmata 4.5–4.7 of [26], we can see that $\tilde{I}^+$ and $\tilde{I}^-$ admit modifications - which we still denote by $\tilde{I}^+$ and $\tilde{I}^-$ - right-continuous, nondecreasing, and $\mathbb{F}$-adapted. Hence, there is also a modification of $\tilde{I}$ - which we still denote by $\tilde{I}$ - belonging to $\mathcal{I}$. Now Fatou's Lemma yields

$$
G(c,d;\tilde{I}) \quad \leq \quad \liminf_{n\to\infty} G(c,d;\tilde{I}^n) \quad = \quad v(c,d), \tag{A.50}
$$

so $\tilde{I}$ is an optimal control starting from (c,d).

Uniqueness. Let $(c,d) \in \mathcal{S}$, and let $I^1 \in \mathcal{I}$, $I^2 \in \mathcal{I}$ be two optimal controls starting from (c,d). Define $\bar{I} := \frac{1}{2}I^1 + \frac{1}{2}I^2$. By linearity of the state equation (2.2) we then have $C^{c,\bar{I}} = \frac{1}{2}C^{c,I^1} + \frac{1}{2}C^{c,I^2}$. Thus, since $g(\cdot,d)$ is convex,

$$
\begin{aligned}
0 \quad \leq \quad G(c,d;\bar{I}) - v(c,d) \quad &= \quad G(c,d;\bar{I}) - \frac{1}{2}G(c,d;I^1) - \frac{1}{2}G(c,d;I^2) \\
= \quad \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(g\Big(\frac{1}{2}C^{c,I^1} + \frac{1}{2}C^{c,I^2}, D_t^d\Big) - \frac{1}{2}g(C_t^{c,I^1}, D_t^d) - \frac{1}{2}g(C_t^{c,I_2}, D_t^d) \right) \right] \quad &\leq \quad 0.
\end{aligned}
$$

So, the inequalities above are indeed equalities and, still due to convexity of $g(\cdot,d)$, we must have

$$
g(C^{c,\bar{I}}, D_t^d) - \frac{1}{2}g(C_t^{c,I^1}, D_t^d) - \frac{1}{2}g(C_t^{c,I_2}, D_t^d) \quad = \quad 0, \quad \mathbb{P} - \text{a.s., for a.e. } t \in \mathbb{R}.
$$

Now the assumption of strict convexity of $g(\cdot, d)$ implies $C^{c,I^1} = C^{c,I^2}$, $\mathbb{P}-$a.s., for a.e. $t \in \mathbb{R}$, from which we derive $I^1 = I^2$, $\mathbb{P}-$a.s., for a.e. $t \in \mathbb{R}$. So, due to right-continuity, I^1 and I^2 are indistinguishable. $\square$

Lemma A.2. *Let $(c, d) \in \mathcal{S}$ and denote by $v_c^+(c, d)$, $v_c^-(c, d)$, respectively, the right- and left-derivative of v w.r.t. c at (c, d) (their existence being guaranteed by convexity of $v(\cdot, d)$). Then*

$$v_c^+(c,d) \quad \leq \quad J(c,d;\sigma,\tau^*), \quad \forall \sigma \in \mathcal{T}; \qquad v_c^-(c,d) \quad \geq \quad J(c,d;\sigma^*,\tau), \quad \forall \tau \in \mathcal{T}. \tag{A.51}$$

Proof. Let us show the first inequality. Let $(c, d) \in \mathcal{S}$ and let $I^* = (I^{*,+}, I^{*,-}) \in \mathcal{I}$ be an optimal control for (c, d). Let $\varepsilon > 0$ and set

$$\tau^* \ := \ \inf\{t \geq 0 \mid I_t^{*,+} > 0\}, \qquad \tau_\varepsilon \ := \ \inf\{t \geq 0 \mid I_t^{*,+} \geq \varepsilon\}.$$

Moreover, given $\sigma \in \mathcal{T}$, set

$$I^\varepsilon \quad := \quad \begin{cases} -I_t^{*,-}, & \text{if } \ 0 \leq t < \sigma \wedge \tau_\varepsilon, \\ I_t^* - \varepsilon, & \text{if } \ t \geq \sigma \wedge \tau_\varepsilon. \end{cases}$$

We can write

$$\begin{aligned} G(c+\varepsilon, d; I^\varepsilon) \quad = \quad & \mathbb{E}\bigg[\int_0^{\sigma\wedge\tau^*} e^{-\rho t} g(c+\varepsilon - I_t^{*,-}, D_t^d)\mathrm{d}t \\ & + \int_{\sigma\wedge\tau^*}^{\sigma\wedge\tau_\varepsilon} e^{-\rho t} g(c+\varepsilon - I_t^{*,-}, D_t^d)\mathrm{d}t + \int_{\sigma\wedge\tau_\varepsilon}^{\infty} e^{-\rho t} g(c - I_t^*, D_t^d)\mathrm{d}t \\ & + \mathbf{1}_{\{\tau_\varepsilon \leq \sigma\}}\big(e^{-\rho\tau_\varepsilon} q_0^+ (I_{\tau_\varepsilon}^{*,+} - \varepsilon) + \int_{\tau_\varepsilon^+}^{\infty} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + \int_0^{\infty} e^{-\rho t} q_0^- \mathrm{d}I_t^{*,-}\big) \\ & + \mathbf{1}_{\{\tau^* \leq \sigma < \tau_\varepsilon\}}\big(e^{-\rho\sigma} q_0^- (\varepsilon - I_\sigma^{*,+}) + \int_{\sigma^+}^{\infty} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + \int_0^{\infty} e^{-\rho t} q_0^- \mathrm{d}I_t^{*,-}\big) \\ & + \mathbf{1}_{\{\sigma < \tau^*\}}\big(e^{-\rho\sigma} q_0^- \varepsilon + \int_{\tau^*}^{\infty} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + \int_0^{\infty} e^{-\rho t} q_0^- \mathrm{d}I_t^{*,-}\big)\bigg], \end{aligned}$$

and

$$\begin{aligned} G(c,d;I^*) = \ & \mathbb{E}\bigg[\int_0^{\sigma\wedge\tau^*} e^{-\rho t} g(c - I_t^{*,-}, D_t^d)\mathrm{d}t + \int_{\sigma\wedge\tau^*}^{\sigma\wedge\tau_\varepsilon} e^{-\rho t} g(c + I_t^*, D_t^d)\mathrm{d}t \\ & + \int_{\sigma\wedge\tau_\varepsilon}^{\infty} e^{-\rho t} g(c + I_t^{*,-}, D_t^d)\mathrm{d}t \\ & + \mathbf{1}_{\{\tau_\varepsilon \leq \sigma\}}\Big(\int_{\tau^*}^{\tau_\varepsilon^-} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + e^{-\rho\tau_\varepsilon} q_0^+ (I_{\tau_\varepsilon}^{*,+} - I_{\tau_\varepsilon^-}^{*,+}) + \int_{\tau_\varepsilon^+}^{\infty} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + \int_0^{\infty} e^{-\rho t} q_0^- \mathrm{d}I_t^{*,-}\Big) \\ & + \mathbf{1}_{\{\tau^* \leq \sigma < \tau_\varepsilon\}}\Big(\int_{\tau^*}^{\sigma^-} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + e^{-\rho\sigma} q_0^- (I_\sigma^{*,+} - I_{\sigma^-}^{*,+}) + \int_{\sigma^+}^{\infty} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + \int_0^{\infty} e^{-\rho t} q_0^- \mathrm{d}I_t^{*,-}\Big) \\ & + \mathbf{1}_{\{\sigma < \tau^*\}}\Big(\int_{\tau^*}^{\infty} e^{-\rho t} q_0^+ \mathrm{d}I_t^{*,+} + \int_0^{\infty} e^{-\rho t} q_0^- \mathrm{d}I_t^{*,-}\Big)\bigg]. \end{aligned}$$

Subtracting we get

$$
\begin{aligned}
v(c+\varepsilon,d)-v(c,d) \quad &\leq \quad \mathbb{E}\Big[\int_0^{\sigma\wedge\tau^*} e^{-\rho t}\big(g(c+\varepsilon-I_t^{*,-},D_t^d)-g(c-I_t^{*,-},D_t^d)\big)\mathrm{d}t \\
&\quad + \int_{\sigma\wedge\tau^*}^{\sigma\wedge\tau_\varepsilon} e^{-\rho t}\big(g(c+\varepsilon-I_t^{*,-},D_t^d)-g(c+I_t^{*,+}-I_t^*,D_t^d)\big)\mathrm{d}t \\
&\quad + \mathbf{1}_{\{\tau_\varepsilon\leq\sigma\}}\Big(e^{-\rho\tau_\varepsilon}q_0^+(I_{\tau_\varepsilon^-}^{*,+}-\varepsilon)-\int_{\tau^*}^{\tau_\varepsilon^-} e^{-\rho t}q_0^+\mathrm{d}I_t^{*,+}\Big) \\
&\quad + \mathbf{1}_{\{\tau^*\leq\sigma<\tau_\varepsilon\}}\Big(e^{-\rho\sigma}q_0^-(I_{\sigma^-}^{*,+}-\varepsilon)-\int_{\tau^*}^{\sigma^-} e^{-\rho t}q_0^+\mathrm{d}I_t^{*,+}\Big)-\mathbf{1}_{\{\sigma<\tau^*\}}e^{-\rho\sigma}q_0^-\varepsilon\Big].
\end{aligned}
$$

Using convexity of $g(\cdot,d)$ we can estimate from above the first two terms in the expectation above respectively with

$$
\varepsilon\int_0^{\sigma\wedge\tau^*} e^{-\rho t}g_c(c-I_t^{*,-},D_t^d)\mathrm{d}t, \qquad L_1(\varepsilon) \ := \ \int_{\sigma\wedge\tau^*}^{\sigma\wedge\tau_\varepsilon} e^{-\rho t}(\varepsilon-I_t^{*,+})g_c(c+\varepsilon,D_t^d)\mathrm{d}t,
$$

while the third term can be rearranged as

$$
-\varepsilon q_0^+e^{-\rho\tau^*}\mathbf{1}_{\{\tau^*<\sigma\}}+L_2(\varepsilon)+L_3(\varepsilon),
$$

where

$$
L_2(\varepsilon) \ := \ \varepsilon q_0^+[e^{-\rho\tau^*}\mathbf{1}_{\{\tau^*<\sigma\}}-e^{-\rho\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon\leq\sigma\}}], \qquad L_3(\varepsilon) \ := \mathbf{1}_{\{\tau_\varepsilon\leq\sigma\}}\Big(e^{-\rho\tau_\varepsilon}I_{\tau_\varepsilon^-}^{*,+}-\int_{\tau^*}^{\tau_\varepsilon^-} e^{-\rho t}\mathrm{d}I_t^{*,+}\Big).
$$

Setting also

$$
L_4(\varepsilon) \ := \ \mathbf{1}_{\{\tau^*\leq\sigma<\tau_\varepsilon\}}\Big(e^{-\rho\sigma}q_0^-(I_{\sigma^-}^{*,+}-\varepsilon)-\int_{\tau^*}^{\sigma^-} e^{-\rho t}q_0^+\mathrm{d}I_t^{*,+}\Big)
$$

we can write

$$
\frac{v(c+\varepsilon,d)-v(c,d)}{\varepsilon} \quad \leq \quad J(c,d;\sigma,\tau^*)+\frac{1}{\varepsilon}\sum_{j=1}^4 L_j(\varepsilon).
$$

Using estimates like the ones used in [28, Lemma 4.3], one can see that, for each $j=1,...,4$, $\frac{L_j(\varepsilon)}{\varepsilon}\to 0$ when $\varepsilon\longrightarrow 0$, which gives the first inequality in (A.51). The second inequality can be obtained in a similar way. □

Proof of Proposition 3.3. Let $v_c^+(c,d)$ and $v_c^-(c,d)$ be, respectively, the left and the right derivative of v w.r.t. c at (c,d), which exist due to convexity of $v(\cdot,d)$ and verify $v_c^+(c,d)\geq v_c^-(c,d)$. Then, considering (A.51), we get

$$
v_c^-(c,d) \quad \leq \quad v_c^+(c,d) \quad \leq \quad J(c,d;\sigma^*,\tau^*) \quad \leq \quad v_c^-(c,d).
$$

So the inequalities above are indeed equalities and hence it follows that $v_c(c,d)$ exists and is equal to $J(c,d;\sigma^*,\tau^*)$. Then, still using (A.51), we get

$$
J(c,d;\sigma^*,\tau) \ \leq \ v_c^-(c,d) \ = \ J(c,d;\sigma^*,\tau^*) \ = \ v_c^+(c,d) \ \leq \ J(c,d;\sigma,\tau^*), \qquad \forall\sigma\in\mathcal{T},\ \forall\tau\in\mathcal{T}.
$$

This shows both the claims. □

Proposition A.4 (Itô's Formula). *Let $\varphi \in C^{1,2}(\mathcal{S};\mathbb{R})$, $(c,d) \in \mathcal{S}$, $I \in \mathcal{I}$, and let τ be a bounded stopping time such that $(C_t^{c,I}, D_t^d)_{t\in[0,\tau]}$ is contained in a compact subset of $\mathcal{S}$. Then the following change of variable's formula holds:*

$$\begin{aligned}\varphi(c,d) &= \mathbb{E}\Big[e^{-\rho\tau}\varphi(C_\tau^{c,I}, D_\tau^d)\Big] + \mathbb{E}\Big[\int_0^\tau e^{-\rho t}[\mathcal{L}\varphi(C_t^{c.I},\cdot)](D_t^d)\mathrm{d}t\Big] \\ &\quad - \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\varphi_c(C_t^{c,I}, D_t^d)\mathrm{d}I_t\Big] \\ &\quad - \mathbb{E}\Big[\sum_{0\le t\le\tau} e^{-\rho t}(\varphi(C_t^{c,I}, D_t^d) - \varphi(C_{t^-}^{c,I}, D_t^d) - \varphi_c(C_t^*, D_t^d)\Delta C_t^{c,I})\Big],\end{aligned}$$

Proof. Theorem 33 (p. 81) in [37] provides the desired formula for functions which are continuously twice differentiable when τ is constant. The extension to the case of τ stopping time for the latter class of functions is standard. To get the formula for functions belonging to $C^{1,2}(\mathcal{S};\mathbb{R})$, one can argue using mollifiers as follows. Take a sequence of mollifiers $(\xi_n)_{n\in\mathbb{N}}$ and consider the convolution $\varphi_n := \xi_n * \varphi$. Then φ_n is continuously twice differentiable for each n, so the formula applies to the sequence $(\varphi_n)_{n\in\mathbb{N}}$. Moreover all the derivatives of φ_n involved in the formula converge locally uniformly to the corresponding derivatives of v (which exist, as the formula involves only derivatives which are defined in the class $C^{1,2}(\mathcal{S};\mathbb{R})$. Hence, the claim follows by uniform convergence since $(C_t^{c,I}, D_t^d)_{t\in[0,\tau]}$ is contained in a compact subset of $\mathcal{S}$. □

Acknowledgement. The authors would like to thank René Aïd and Bertrand Villeneuve for useful discussions. Salvatore Federico also thanks Giorgio Ferrari for comments on the literature and for suggesting the possible extension of the model described in Remark 2.4.

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